STATISTICAL LAWS OF DEMAND AND SUPPLY

HENRY SCHULTZ

MATERIALS FOR THE STUDY OF BUSINESS

STATISTICAL LAWS OF DEMAND AND SUPPLY

WITH SPECIAL APPLICATION TO SUGAR

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PREFACE

The first economists who studied the law of supply and demand lived in an age when markets were less extensive, more distant from one another, and much more independent of one another than they are today. This is reflected in the early theories of supply and demand, which are depositories of the experience of the late eighteenth- and early nineteenth-century economists with comparatively simple market phenomena more or less vaguely represented by words.

With the improvements in the means of transportation and communication and with the growing general complexity of our economic life markets have become more extensive and more dependent upon one another than they have ever been before, and market phenomena have become exceedingly complex. As a consequence, the advantages which the earlier theories enjoyed by reason of their simplicity and freedom from mathematical symbols were now outweighted by the disadvantages which attached to their incompleteness and vagueness. Economists began to see that they really said nothing at all when they stated that "the price of goods is in the inverse ratio of the quantity offered, and in the direct ratio of the quantity demanded." They also began to perceive that prices formed an interrelated, interconnected system, and that to say that the quantity of a commodity demanded is a function only of its price and not of a good many other (theoretically *all* other) prices is to adopt an oversimplification which does not give an insight into the extreme complexity of the price system. This state of affairs clearly called for a better type of economic theory.

The more complete and scientific type of theory is due chiefly to the genius of Leon Walras, who, building on Cournot's epoch-making work, first succeeded, by the use of simultaneous equations, in completely surveying the interrelated factors in ex-

change, production, and distribution in such a way as to give a truer insight into the complexity of our economic system and in paving the way for subsequent applications of the quantitative method to the problem of measuring the relative strength of the different economic forces. In this work Walras was followed by Pareto, who generalized his master's system and improved upon it.

The epoch-making discoveries of Walras and Pareto must not, however, blind us to the fact that their theory is a theory of static equilibrium. Dynamic changes that take place from time to time are not taken into account by this theory. (And no one is more aware of this than the mathematicians themselves.) But it is a truism to say that economic phenomena are not static, but dynamic. In order, therefore, to bring the general theory of equilibrium into closer agreement with the facts of our economic experience, it is necessary to pass from a statical equilibrium to a moving equilibrium. This problem has two aspects: the purely deductive or mathematical, and the statistical. The first has not as yet been successfully studied, although the signs point to the possibility of the development of a mathematical theory of dynamical economics in the next few decades. The second has recently been solved by Professor Henry Ludwell Moore, of Columbia University, who was also the first to derive statistical laws of demand and supply, in his paper on "A Theory of Economic Oscillations" in the *Quarterly Journal of Economics* for November, 1926.

In Professor Moore's solution of the problem of general equilibrium "the equations expressing the relation between the parts of the economic system receive the definite, numerical form in which theory admits of empirical testing." The solution of the problem of general equilibrium may thus be "expressed in terms that admit of immediate practical application."

This advance would not, however, have been possible without the aid afforded by the more modern methods of statistical analysis. It is these methods which enable the statistical econo-

mist to segregate the important factors of a given economic situation, to measure their relative strength, and to ascertain the laws according to which they produce their individual and joint effects. It is these methods which enable him to study the shifting of the economic equilibrium from time to time as a result of dynamic changes. It is, indeed, a fortunate occurrence that not many years after the originators of the modern type of economic theory laid the foundations of their system, biometricians and mathematicians began to develop the newer body of statistical technique, without which economics would lose much of its practical utility.

This book is an attempt to apply some of the methods of modern statistics to the derivation of two functions which arise in almost any discussion of economic equilibrium—the demand function and the supply function. More specifically, it is the object of this book to examine some of the difficulties which arise in the derivation of statistical laws of demand and supply, to indicate how these difficulties may be overcome, and, by way of illustration, to derive the demand and supply curves for sugar, to show how the equilibrium of demand and supply changes from time to time, and to indicate the bearing of the results on the tariff on sugar.

Though statistical demand and supply curves can be best understood when they are examined from the point of view of the general equilibrium theory, it must not be assumed that they are, therefore, of no importance to the adherents of some of the other schools of economic thought. The fact is that a demand "curve" or a supply "curve" is, and ought to be, of much greater importance to the neo-classicist than to the adherent of the theory of general equilibrium, for the latter is compelled always to keep in mind that demand and supply functions are only two out of the several functions that are necessary in the determination of the general economic equilibrium.

Part I of this book, which relates to demand, is a reprint, with some important additions and changes, of studies which ap-

peared in the *Journal of Political Economy* for October and December, 1925 (Vol. XXXIII, Nos. 5 and 6). Chapter iv of Part II, which deals with some of the theoretical considerations relating to supply, also first appeared in the *Journal of Political Economy* (Vol. XXV, No. 4, August, 1927). The rest consists of entirely new material, although I had occasion to draw on it in a paper which I read at the Seventeenth Annual Meeting of the American Farm Economic Association at St. Louis in December, 1924, entitled, "Cost of Production, Supply and Demand, and the Tariff," and which was published in the *Journal of Farm Economics* for April, 1927.

As he examines the conclusions reached in this book the reader will do well to keep in mind the fact that the *results* of a method cannot be separated from the method itself. The fine differences between the various methods—differences which are not always apparent and which cannot always be conveniently explained—produce differences in the results obtained with their aid. Thus the values of the elasticities of demand and supply derived in this book depend to a large degree on the particular method of curve-fitting employed. Had the common method of curve-fitting been adopted, the results would have been quite different.

In the preparation of this book the aim has been to throw some light on certain problems in statistical economics, and not to provide a method for forecasting the price of sugar. It is to be hoped, however, that the competent trade statistician and economist will find both the methods and the conclusions of this work not without practical importance, and that this work will make it easier for him to develop such second and third approximations as he may find necessary.

ACKNOWLEDGMENTS

The investigations on which this book is based were begun some five years ago and have put me under an increasing sense of indebtedness to many persons who have lent their assistance.

Professor Henry L. Moore, of Columbia University, not only offered keen and sympathetic criticism of Part I and gave generous help with the proofs, but also aided with inspiration, advice, and criticism during the entire preparation of that part of the book. Mr. Philip G. Wright and Mr. Isador Lubin, of the Institute of Economics, read the manuscript of Part I and made valuable suggestions. Professor Wesley C. Mitchell, of Columbia University, read the proofs of Part I and gave much useful counsel. Mrs. Arcadia Near Phillips, formerly of the Institute of Economics, and Miss Myrtle Opdyke, of the United States Tariff Commission, assisted with some of the graphs and computations. Mrs. Ardis T. Monk and Miss Ramona Simpson assisted with the final proofs.

Professor Frederick C. Mills, of Columbia University read the entire work in manuscript form and the proofs of Part II, and gave most helpful, close, and penetrating criticism. My colleague, Professor Jacob Viner, also read the manuscript of the entire book and kindly and freely gave advice, opinions, and criticisms which have been of the greatest service. To the other authorities who helped on specific questions specific acknowledgment is made in footnotes. To Professor Leon C. Marshall, of the University of Chicago, a special debt of gratitude is due for his interest in this work and for having rendered possible its publication.

The various criticisms and suggestions which I was fortunate enough to receive have led to the removal of some too unqualified statements, some confused exposition, and some positive mistakes. I cannot claim, however, to have satisfied all of my critics, and I must, therefore, assume full responsibility for all errors of fact and inaccuracies and inadequacies of analysis.

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PART I. THE STATISTICAL LAW OF DEMAND

CHAPTER I

THEORETICAL CONSIDERATIONS RELATING TO DEMAND

I. INTRODUCTION

"The disturbance of the conditions of supply by a tax or a bounty . . . .," says Professor Edgeworth, "gives rise to problems too complicated for the unaided intellect to deal with." Certain it is that the popular method of attacking these problems is erroneous. This is evident as soon as one asks, for example, "What is the relation between the tariff on sugar and the prices paid by the consumer?" The popular method of deriving an answer to this question consists essentially of drawing up tables or graphs showing that prices, after the imposition of the tariff, have risen (or fallen). Of course this proves nothing with respect to the effect of the tariff on prices, because it does not eliminate the *other causes* which have contributed to the price changes. What we desire to know is, not how much higher prices were after the imposition of the tariff than they were before, but how much of the increase is due to the tariff, and how much to other causes. This question may be quite definitely (i.e., quantitatively) answered if we know the coefficients of the elasticities of the domestic and foreign demand and supply.[1]

The coefficients of the elasticities of demand and supply also enter as factors into the solution of the related problems of the effect of the tariff on consumption, on imports, on the profits of the domestic producers, and on the revenue derived from the duty. These coefficients are also of major importance in the

[1] A. C. Pigou, "The Known and Unknown in Mr. Chamberlain's Policy," *Fortnightly Review* (January, 1904), p. 44; *Protective and Preferential Import Duties* (1906), pp. 94–95; *Economics of Welfare* (1920), p. 942; H. Schultz, "The Statistical Measurement of the Elasticity of Demand for Beef," *Journal of Farm Economics*, June, 1924.

theory of monopoly profits, and in problems connected with price-fixing. In fact, there are many problems in applied economics which remain unsolved only because the numerical values of these constants are not known.

The object of Part I is to examine some of the difficulties which arise in the derivation of the elasticity of demand, to indicate how these difficulties may be overcome, and to derive the elasticity of demand for sugar.

II. THE CONCEPT OF ELASTICITY OF DEMAND

The coefficient of elasticity may be defined as the proportion by which a given relative change in price alters the quantity demanded or supplied, when the relative changes are small. Hence the coefficient of the elasticity of demand may be defined as the ratio of the relative change in the quantity demanded to the corresponding relative change in price, when the relative changes are infinitesimal. In Professor Marshall's words:

> The *elasticity* (or *responsiveness*) *of demand* in a market is great or small according as the amount demanded increases much or little for a given fall in price, and diminishes much or little for a given rise in price. We may say that the elasticity of demand is one, if a small fall in price will cause an equal proportionate increase in the amount demanded: or as we may say roughly, if a fall of one per cent. in price will increase the sales by one per cent.; that it is two or a half, if a fall of one per cent. in price makes an increase of two or one half per cent. respectively in the amount demanded; and so on. (This statement is rough because 98 does not bear exactly the same proportion to 100 as 100 does to 102.)[1a]

In mathematical symbols, the coefficient of elasticity of demand, η, is

$$\eta=\frac{dx}{x}\bigg/\frac{dy}{y}=\frac{d\log x}{d\log y}=\frac{dx}{dy}\cdot\frac{y}{x} \tag{1}$$

where x is the quantity demanded and y the price per unit. If this is numerically equal to 1.0 for all values of x, the demand is neither elastic nor inelastic. If this is numerically greater than

[1a] Alfred Marshall, *Principles of Economics* (8th Edition), p. 102 and note.

1.0, the demand is said to be *elastic*. If it is numerically less than 1.0, the demand is said to be *inelastic*. The size of the coefficient measures the degree of elasticity.

The term *numerically* is introduced in order to give the concept of elasticity the greatest possible generality of definition, i.e., in order to make the same definition cover the presumably rare positively inclined demand curve as well as to the more common negatively inclined curve.[2] Marshall confines his discussion to the latter, thus getting a negative value for η, which he changes into a positive value by giving it a negative sign. We have not followed this practice in our definition.

For the well-known negatively sloping demand curve, the concept of elasticity of demand may be given an interesting geometrical interpretation. If $\eta = -1.0$ for all values of x, that is, if the demand is neither elastic nor inelastic, but a "constant outlay demand," the curve which represents it—"a constant outlay curve"—is a rectangular hyperbola. If it is *less* than -1.0 (elastic demand), the curve which represents it cuts the rectangular hyperbola from below. If it is *greater* than -1.0 (inelastic demand), the curve which represents it cuts the rectangular hyperbola from above.[3]

For measuring the elasticity of positively sloping demand curves, i.e., of curves which represent an increase in price as accompanied by an increase in the quantity demanded, the "constant outlay curve" cannot serve as a criterion, for constancy of outlay under these conditions is impossible in the nature of the

[2] For discussions of the possibility of obtaining demand curves of the positive type, see Vilfredo Pareto, *Manuel d'Économie Politique* (1909), pp. 579–91, and his paper on "Économie Mathématique," pp. 628–31, in *Encyclopédie des Sciences Mathématiques,* Tome I, Vol. IV, Fascicule 4. Antonio Osorio, *Théorie Mathématique de L'Échange* (1913), pp. 285–95. Wl. Zawadzki, *Les Mathématiques Appliquees à L'Économie Politique* (1914), p. 186. Henry Ludwell Moore, *Economic Cycles* (1914), pp. 110–16; and *Forecasting the Yield and the Price of Cotton* (1917), p. 150.

[3] Cf. Marshall, *op. cit.,* Mathematical Appendix, pp. 839–40; Henry Ludwell Moore, "Elasticity of Demand and Flexibility of Prices," *Journal of the American Statistical Association,* March, 1922.

case. Of this family of curves, the one that has unit elasticity throughout its entire extent is the straight line which passes through the origin. The *slope* of such a line has no effect on the value of the elasticity of demand.[4] This property of the straight line is, however, of greater importance in connection with the study of *supply* than with demand.

The concept of elasticity of demand is due to Cournot, who (in 1838) defined it in mathematical symbols without troubling himself to give it a name.[5]

Cournot deduced the condition for maximizing total receipts, xy, and suggested that "Commercial statistics should be required to separate articles of high economic importance into two categories, according as their current prices are above or below the value which makes a maximum of $pF(p)$ [our yx]. We shall see that many economic problems have different solutions, according as the article in question belongs to one or the other of these two categories" (p. 54). But the condition for this maximum is identical with that for an elasticity of demand of -1.0; or, that point on the (negatively inclined) demand curve whose co-ordinates inclose the maximum area is also the one at which the elasticity of demand is $\eta = -1.0$; so that Cournot's suggestion is really to the effect that commodities should be classified according to their elasticities of demand. Marshall gave the formula $\eta = \frac{dx}{x} \bigg/ \frac{dy}{y}$ a name, and popularized it. But the concept,

[4] If $\eta = \frac{dx}{x} \big/ \frac{dy}{y} = +1.0$, by hypothesis, it also follows that $\frac{dy}{y} = \frac{dx}{x}$.
Integrating the last expression, we have

$$\log y = \log x + \log b$$

or

$$y = bx,$$

which is the equation to a straight line passing through the origin. Hence, all positively sloping straight lines passing through the origin have an elasticity of $+1.0$. It is also worth noting that for such lines there is no distinction between "point" elasticity and "arc" elasticity.

[5] Augustin Cournot, *Mathematical Principles of the Theory of Wealth* (Bacon's translation), pp. 52–54.

which Mr. Keynes considers as one of Marshall's seven more striking contributions to knowledge, "without the aid of which the advanced theory of value and distribution can scarcely make progress," is not "wholly Marshall's," as Keynes has been led to believe.[6]

In connection with the definition of the elasticity of demand given before two points must be emphasized.

1. The definition presupposes a knowledge of the demand curve. Consequently, any attempt to derive the coefficient of the elasticity of demand for a commodity without first deriving the equation to the demand curve is apt to lead to difficulties.[7]

2. The coefficient of elasticity relates to *a point* on the demand curve, and may vary in magnitude from point to point. In giving *the* coefficient of elasticity of a commodity, one must, therefore, specify the point on the demand curve to which it applies, unless, of course, the coefficient is the same at every point on the curve.[8]

Marshall gives (p. 102, note) the following rule for determining graphically the coefficient of elasticity at any point (see Fig. 1):

> Let a straight line touching the curve at any point, P, meet OX in T and OY in t, then *the measure of elasticity at the point P is the ratio of PT to Pt.*
>
> If PT were twice Pt, a fall of 1 per cent. in price would cause an increase of 2 per cent. in the amount demanded; the elasticity of demand

[6] John Maynard Keynes, "Alfred Marshall, 1842–1924," *Economic Journal,* XXXIV (September, 1924), 353. Marshall must have completely forgotten Cournot's contribution to this subject by the time he began to write his own famous chapter on demand, for it is inconceivable that a man of his nobility of character would wittingly fail to do justice to any author, and especially to Cournot, of whom he wrote: "Cournot's genius must give a new mental activity to everyone who passes through his hand."

[7] Compare Professor Lehfeldt's attempt to derive the elasticity of demand for wheat (*Economic Journal,* 1914, pp. 212 ff.), of which more later.

[8] The equation to a demand curve representing a constant elasticity, n, at every point is $xy^n = c$. See Marshall, *op. cit.,* p. 840.

would be two. If PT were one-third of Pt, a fall of 1 per cent. in price would cause an increase of ⅓ per cent. in the amount demanded; the elasticity of demand would be one-third; and so on.[9]

The objection is sometimes raised that the elasticity *at a point* on a curve, corresponding, as it does, to *infinitesimal*

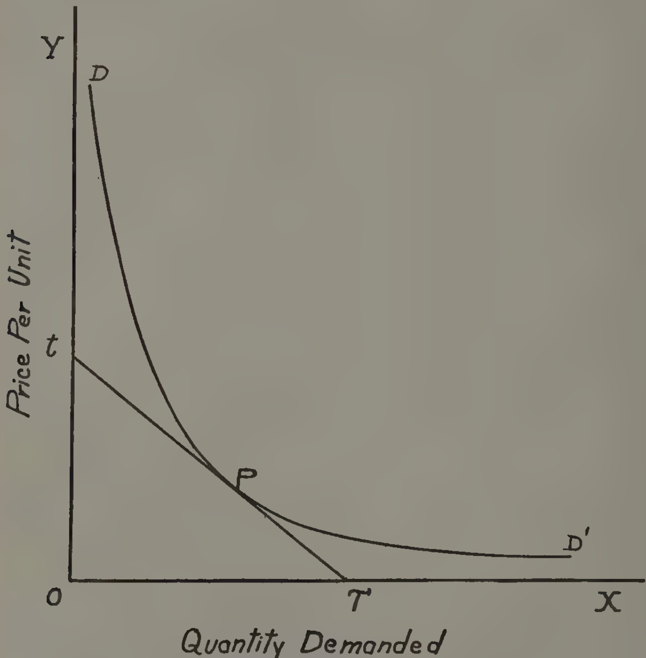

FIG. 1.—Geometrical measure of the elasticity of demand
$\eta = TP/Pt$

changes in the quantity demanded and in the demand price, *"tells us nothing"* of the elasticity corresponding to *finite*

[9] For proof, see *ibid.*, Mathematical Appendix, Note III. This rule is also contained, in somewhat different form, in Cournot, *op. cit.*, p. 54 and Figure 1 of that work.

changes. This objection has been elaborated by Professor Dalton,[10] who claims that the distinction between elasticity at a point and elasticity across a finite arc, or, as he terms it, between "point elasticity" and "arc elasticity," is of practical importance.

The distinction may be made clear by reference to a numerical illustration.

Suppose a "constant outlay demand curve," such that, whatever the price of the commodity, \$1,200 worth of it will be sold (Fig. 2). The equation to this curve is $xy = \$1{,}200$, where x is the quantity sold and y is the price per unit. We have seen before that so long as we confine ourselves to infinitesimal changes, the elasticity at *every point* on such a curve is equal to one. Furthermore, as this is a constant-outlay curve, "any (small) fall in price will cause a proportionate increase in the amount bought."[11]

Let us now measure the elasticities corresponding to *finite* changes. In other words, let us determine the elasticities corresponding to various *arcs* (not points) on the demand curve. From the equation of the demand curve we know that when the price per unit is \$4.00, the quantity demanded is 300 units. If the price rises from \$4.00 to \$6.00, or by 50 per cent, the quantity demanded drops from 300 to 200, or by 33⅓ per cent, and the elasticity of demand is $-\frac{33\frac{1}{3}\%}{50\%} = -\frac{2}{3}$. If, on the other hand, the price falls from \$4.00 to \$2.00, or by 50 per cent, the amount demanded rises from 300 to 600, or by 100 per cent, and the elasticity of demand is $-\frac{100\%}{50\%} = -2$. Now let the price return to its former level, that is, let the price rise from \$2.00 to \$4.00, or by 100 per cent. The quantity demanded will fall from 600 to 300, or by 50 per cent. Here the elasticity of demand is

[10] Hugh Dalton, *Some Aspects of the Inequality of Incomes in Modern Communities* (1920), pp. 192–97.

[11] Marshall, *op. cit.*, p. 839.

$-\frac{50\%}{100\%} = -\frac{1}{2}$. It will be seen from the foregoing examples that, although the *point* elasticity of this curve is always -1, the *arc* elasticity is never equal to -1; that is, a given *finite* per-

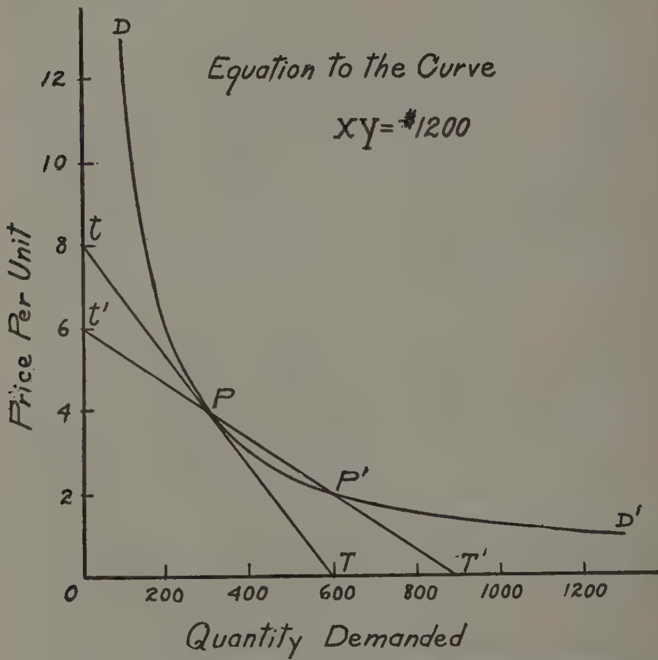

FIG. 2.—Geometrical difference between "point" elasticity of demand and "arc" elasticity of demand.

"Point" elasticity of demand at $P = TP/Pt$

"Arc" elasticity of demand (with base P) $= T'P/Pt'$

"Arc" elasticity of demand (with base P') $= T'P'/P't'$

centage change in price does *not* cause an equal percentage change in the amount demanded. Furthermore, the magnitude of the (arc) elasticity depends upon the particular end of the arc

which is taken as a base, or—which is the same thing—upon the *direction* in which the price is assumed to change. It is always numerically greater than 1 for a fall in price and an increase in demand, and is always numerically less than 1 for a rise in price and a decrease in demand.[12]

In view of these differences between point elasticity and arc elasticity, Professor Dalton claims that Marshall's characterization of a constant-outlay curve "contains a definite mathematical error." The particular passage in Marshall's *Principles* to which he takes such vigorous exception is: "If the elasticity of demand be equal to unity for all prices of the commodity, any fall in price will cause a *proportionate*[13] increase in the amount bought, and therefore make no change in the total outlay which purchasers make for the commodity" (p. 839).

Now while it may be freely granted that there is a valid and useful distinction between point elasticity and arc elasticity, and that those formulas in economics which are based on the assumption of small changes in the variables under consideration ought not to be applied to problems in which the changes are large, we cannot go so far as to agree that the passage just quoted "contains a definite mathematical error." There really is no difference between the point elasticity and the arc elasticity of a constant-outlay curve (or of any other curve having a constant coefficient of elasticity), provided the elasticity is satisfactorily measured. The characteristic of a constant-outlay curve is that any change in price causes a *proportionate* change in the amount bought. Dalton assumes that proportionate changes are best measured by *percentage* increases or decreases, and by making the percentage changes large enough, as in the three arithmetical examples presented above, he obtains widely varying values for his (arc) coefficient of elasticity. But percentages are not good measures of proportionate changes when the bases on which they are computed are not kept constant. Thus a com-

[12] For a general proof of the foregoing see Hugh Dalton, *op. cit.*, p. 195.

[13] Italics are mine.

modity that quadrupled in price and then receded to its former level would show a 300 per cent rise and a 75 per cent drop. A much better measure of proportionate changes is to be had by taking the difference between the *logarithms* of the numbers. The coefficient of elasticity is then simply the ratio of the difference between the logarithms of the quantities to the difference between the logarithms of the prices. When the changes in both quantities and prices are measured *logarithmically,* there is no difference whatsoever between the point elasticity and the arc elasticity of the constant-outlay curve. That the use of logarithms is not only permissible, but almost mandatory, follows at once from the mathematical definition of the coefficient of elasticity given above, namely,

$$\eta = \frac{dx}{x} \bigg/ \frac{dy}{y} = \frac{d \log x}{d \log y}.$$

If logarithms are used instead of simple numbers, the demand curve DD' in Figure 2 becomes a straight line whose *slope* measures the coefficient of elasticity.[14] Obviously, it makes no difference whether one measures the coefficient of elasticity, i.e., the slope of the line, by finding the ratio of the base to the altitude of a small triangle (point elasticity) or of a large triangle (arc elasticity). The ratio is the same in both cases.

Even if the equation to the demand curve is such that it cannot be reduced to a linear form by logarithmic transformation, it is still more desirable to measure (arc) elasticity by the logarithmic method than by the arithmetic (percentage-change) method. The latter, as was pointed out above, is open to the serious objection that it varies according to the *direction* in which the percentage changes occur. Thus, in the case of the

[14] The equation to the constant-outlay curve is $xy = c$. Taking logarithms, this becomes $\log x + \log y = \log c$, which may be written $\log y = -\log x + \log c$. This, of course, is the equation to a straight line having a slope of -1. By definition,

$$\eta = \frac{dx}{x} \bigg/ \frac{dy}{y} = \frac{d \log x}{d \log y} = -1.$$

constant-outlay curve given above, the arc elasticity (arithmetically measured) corresponding to a fall in price from \$4.00 to \$2.00 per unit is -2, while the corresponding elasticity for a rise in price from \$2.00 to \$4.00 is $-\frac{1}{2}$. This objection is recognized by Dalton himself, but he claims that the difference between these two elasticities "will not generally be large" (p. 195). This claim, however, cannot be substantiated. Dalton's entire argument as to the distinction between point elasticity and arc elasticity is based on the assumption that the changes in the prices (or in the quantities) are large; for, if they are small, arc elasticity even when measured by the arithmetic method is the same as point elasticity for all practical purposes. But when the changes in the variables are large, it does make a difference—and a big one—whether the arc elasticity is measured (arithmetically) from the lower end of the arc or from the upper end.[15]

Basically the difference between arc elasticity and point

[15] It might appear from the foregoing discussion that too much emphasis is being placed on such elementary notions as the elasticity of demand and the limitations of percentage changes as measures of relative changes, were it not for the fact that lack of familiarity with these notions is sometimes responsible for fallacious economic reasoning. Thus, there are not a few economists who contend that, if the price of a commodity rose by, say, 100 per cent while its supply declined by, say, only 33⅓ per cent, the decrease in the supply *could not possibly* explain the *entire* increase in the price, but that other causes—decline in the purchasing power of money, etc.—must have contributed to the price increase. (Presumably only one-third of the increase can be attributed to the contraction of the supply, while two-thirds must be attributed to other causes!)

What these economists overlook, of course, is that a large or small difference between a percentage change in supply and a corresponding percentage change in price proves absolutely nothing with regard to the existence or non-existence of "other causes." Even under static conditions where "all other things remain equal" by hypothesis, a given percentage change in supply may bring about almost any imaginable percentage change in price, especially when the percentage changes are large.

In Figure 2 for example, there is traced a demand curve with a coefficient of elasticity of demand of -1 at every point on the curve. That is to say, a 1 per cent reduction in the supply always brings about a 1 per cent (approximately) increase in price. In the same curve, however, a 50 per cent reduction in the supply brings about a 100 per cent increase in price!

elasticity is simply a difference between finite differences and infinitesimals. The prices and quantities which go to make up any demand schedule proceed, of course, by finite differences, which are frequently very large. In the words of Professor Edgeworth, they "may be likened to a net with meshes so wide as to lose half the catch." From such data we can derive arc elasticity—which may or may not have any significance—but no point elasticity. When, however, we "fill up the vacant spaces with a finer reticulation" by passing a continuous curve through the points, we can derive from it both the point elasticity and the arc elasticity, and judge the significance of the latter. The thing to remember, therefore, is that in order to derive an unequivocal coefficient of elasticity we must first derive the equation to the demand curve, or the law of demand. To this we now turn.

III. THE NEO-CLASSICAL LAW OF DEMAND

The concept of demand which is most useful to the statistical economist is that which was first propounded by Cournot nearly a century ago. Brushing aside such meaningless and sterile statements as "the price of goods is in the inverse ratio of the quantity offered, and in the direct ratio of the quantity demanded," Cournot stated the concept of demand in the following unambiguous terms: "Let us admit," said he, ". . . . that the sales or the annual demand D is, for each article, a particular function $F(p)$ of the price p of such article. To know the form of this function would be to know what we call the *law of demand* or *of sales*."[16] Some of the essential features of this law may be summarized as follows:

1. The function $F(p)$ which expresses the law of demand or of the market is assumed to be a *continuous* function. That it might be otherwise if the number of consumers were very limited is clearly recognized. But the economist is not interested so much in the individual demand functions as in the

[16] Cournot, *op. cit.*, p. 47.

corresponding aggregate or collective function. The latter tends to be continuous.

The statistical conclusion to be drawn from this is that the observations must be numerous.

2. The law of demand or the demand curve relates to a given country or market. "It is well known that by *market* economists mean, not a certain place where purchases and sales are carried on, but the entire territory of which the parts are so united by the relations of unrestricted commerce that prices there take the same level throughout, with ease and rapidity."[17]

In any statistical investigation of the demand for a commodity, the market to which it relates must be specified. The exact boundaries of a market are, however, sometimes difficult to determine. In this connection Professor F. A. Fetter's "Economic Law of Market Areas"[18] may be helpful. In his own words:

> The boundary line between the territories tributary to two geographically competing markets for like goods is a hyperbolic curve. At each point on this line the difference between freights from the two markets is just equal to the difference between the market prices, whereas on either side of this line the freight difference and the price difference are unequal. The relation of prices in the two markets determines the location of the boundary line: the lower the relative price the larger the tributary area [p. 525].

3. The law of demand relates to a *point* in time. In most statistical investigations, however, the year is taken as the natural unit of time; that is, the quantities studied are those which are sold *annually*, and the prices with which the quantities are compared are *average annual* prices. But, as Cournot pointed out, both the price and the law of demand of a com-

[17] *Ibid.*, p. 51, footnote.

[18] *Quarterly Journal of Economics*, May, 1924.

This law, which Professor Fetter discovered and found useful in connection with his work on the "Pittsburgh-plus" plan before the Federal Trade Commission, may be traced to von Thünen's *Der isolirte Staat.* See Marshall, *Principles of Economics* (8th ed.), p. 442, note.

modity may vary considerably during the year.[19] This raises some interesting questions: Is it possible for a commodity to have an *elastic* demand during one season and an *inelastic* demand during another? Suppose that monthly data of consumption (or production) and prices yield an elastic (or inelastic) demand. Would annual averages derived from the same data also yield an elastic (or inelastic) demand? It is to be hoped that statistical data will be forthcoming which will enable one to derive specific answers to these questions.

Should the available data for any particular commodity enable one to derive seasonal demand curves as well as the average of such curves, that curve will, of course, be chosen which, for the purposes of the investigation in question, is the most significant.

4. The demand curve may represent price either as a function of the amount demanded or "consumed" by the buyers (non-possessors) only, or as a function of the amount demanded by both buyers and sellers (non-possessors and possessors). Thus, a demand curve for potatoes may represent either (1) the "amounts which will be bought" at each of a variety of prices, or (2) the *total* amount which will be "taken" at each price, which includes not only the amount bought by "consumers," but also the amount kept by farmers for their own use and for feeding, as well as the amount left in the ground in case the price obtainable does not pay the costs of harvesting.[20] The first is the more common concept of the demand curve. For some purposes, however, the second concept is more useful.

The *rationale* of the second or more general concept of

[19] Cournot, *op. cit.*, p. 52. The author is quite satisfied that the elasticity of demand for beef varies greatly between summer and winter. See "The Measurement of the Elasticity of Demand for Beef," *Journal of Farm Economics*, July, 1924, pp. 275–76.

[20] Cf. Holbrook Working, "Factors Determining the Price of Potatoes in St. Paul and Minneapolis," *Technical Bulletin 10, University of Minnesota Agricultural Experiment Station* (October, 1922), p. 17.

demand is clearly stated by Professor Davenport in his *Economics of Enterprise*:

> In any market where owners of goods will sell only if they can "get their price," and owners of money buy only if they can "get their money's worth," it is evident that the men on both sides of the market have made an appraisal of the thing *owned* relatively to the thing *offered.* This fact points to the similarity between the demand and supply side of any particular exchange; the owners of hats may be thought of as having either a supply of hats or a demand for money,—the owners of money as having either a demand for hats or a supply of money. Each side has things for exchange, with limits on exchange expressed in the other thing.
>
> Because of this similarity it is possible to combine the buyers' and sellers' money estimates of hats into one schedule which will express for each man the relation for him between money and hats. . . . (pp. 48–51).
>
> It is . . . . evident that if we decide to regard the money side of the situation as demand for hats, and the hat side of the situation, not as demand for money, but only as supply of hats, we must recognize the holders of hats as themselves having demands for hats (p. 51).

And again,

> . . . . the reservation prices of the sellers are, in ultimate analysis, demands, and are as important to the fixation of price, and important in precisely the same way, as are the price paying dispositions of the seekers for goods (p. 55).

Another strong champion of this concept is P. H. Wicksteed. Wicksteed takes up the question of the rôle which the supply curve plays in the determination of price in the following words:

> But what about the "supply curve" that usually figures as a determinant of price, co-ordinate with the demand curve? I say it boldly and baldly: There is no such thing. When we are speaking of a marketable commodity, what is usually called a supply curve is in reality a demand curve of those who possess the commodity; for it shows the exact place which every successive unit of the commodity holds in their relative scale of estimates. The so-called supply curve, therefore, is simply a part of the total demand curve. . . . .[21]

[21] "The Scope and Method of Political Economy in the Light of the 'Marginal' Theory of Value and Distribution," *Economic Journal,* March, 1914, p. 13.

It must be emphasized, however, that the insignificant state to which these economists would reduce the supply curve relates only to its generally accepted rôle as a factor in price determination co-ordinate with the demand curve. If all that we desire to know is the *price* at which a given exchangeable commodity will sell in the market, the "supply curve" is not necessary. The price is fixed absolutely by two determinants: (1) the general or total demand curve (of which the "supply curve" is a part), and (2) the amount of the actual supply existing in the community. "This is not a curve at all, but an actual quantity." If, however, what we wish to know is not simply the price at which the commodity will be sold, but also the number of units that will change hands, and whether the price is in stable or unstable equilibrium, we must have recourse to the ordinary method of presenting the general demand curve in two sections: the ordinary conventional demand curve of the purchasers, and supply curve of the sellers.

That these are valid conclusions may be illustrated by reference to a hypothetical example.[22] Suppose that in a given market for hats there are five buyers, B_1, B_2, B_3, B_4, and B_5, and

TABLE I*

B_1 is willing to buy one hat for $5.00	
B_2 is willing to buy one hat for $4.00	S_1 is willing to sell one hat for $4.00
B_3 is willing to buy one hat for $3.00	S_2 is willing to sell one hat for $3.00
B_4 is willing to buy one hat for $2.00	S_3 is willing to sell one hat for $2.00
B_5 is willing to buy one hat for $1.00	S_4 is willing to sell one hat for $1.00

* Professor Henry L. Moore calls attention to the implicit assumption made in this example, i.e., that each seller has only one unit to sell and each buyer will take only one unit. This purely hypothetical assumption does not, however, invalidate the use of the example for the purpose of illustrating how the general demand schedule may be derived from the buyers' and sellers' schedules.

four sellers, S_1, S_2, S_3, and S_4, and that the situation is as shown in Table I.

Under these conditions, what will the price be?

It is evident that if B_1 is willing to buy one hat for $5.00. he certainly would be willing to buy it at a price less than $5.00,

[22] Cf. H. J. Davenport, *Economics of Enterprise*, pp. 47–51.

so that when the price is \$4.00, the number of hats demanded is two (one by B_1 and one by B_2); when the price is \$3.00, the number of hats demanded is three (one by B_1, one by B_2, and one by B_3); etc. Similarly, if S_4 is willing to sell one hat for \$1.00, he certainly would be willing to sell it at a higher price, so that when the price is \$2.00, two hats are offered on the market (one by S_4 and one by S_3); when the price is \$3.00, three hats are offered on the market (one by S_4, one by S_3, and one by S_2); etc. Proceeding in this manner, we obtain the demand and supply schedules shown in Table II.

TABLE II

Price	Quantity Demanded	Quantity Supplied
\$5.00	1	....
4.00	2	4
3.00	3	3
2.00	4	2
1.00	5	1

It is clear from these schedules that the market price will be \$3.00, for at this price the number of hats demanded is equal to the number supplied. As there are four hats on the market, one hat will not find a purchaser. (S_1 is not willing to sell his hat for less than \$4.00.)

When these schedules are represented graphically, we have the usual demand and supply curves. In Figure 3, DD' is the demand curve of the buyers and SS' is the supply curve of the sellers. The intersection of the two curves is at the point P. From the co-ordinates of this point ($X=3$, $Y=3$) we know that three hats will be sold and that the price will be \$3.00 per hat. From the *positive slope* of the supply curve SS' we know that the demand and supply are at stable equilibrium at P.

Now let us combine both buyers' and sellers' schedules into one general demand schedule. Since the owners of hats may be thought of as having either a supply of hats or a demand for money, and the owners of money as having either a demand for

hats or a supply of money, the market situation may be represented as shown in Table III.

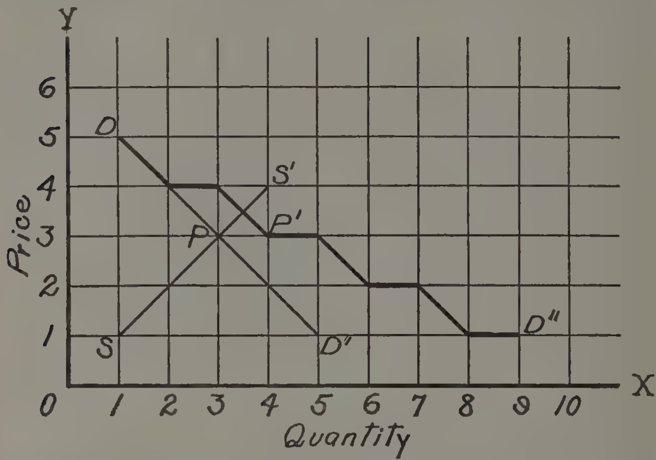

FIG. 3.—The relation between the general demand curve and the particular demand and supply curves.

In Figure 3 this general demand schedule is represented graphically by the curve *DP′ D″*. Thus B_1's estimate is represented by the point 1,5; B_2's estimate by the point 2,4; S_1's estimate by the point 3,4; etc.

TABLE III

Those Who Have Hats	Those Who Have Money	
	B_1.......	estimates 5 of money as equal to 1 hat
	B_2.......	estimates 4 of money as equal to 1 hat
S_1............	....	estimates 4 of money as equal to 1 hat
	B_3.......	estimates 3 of money as equal to 1 hat
S_2............	....	estimates 3 of money as equal to 1 hat
	B_4.......	estimates 2 of money as equal to 1 hat
S_3............	....	estimates 2 of money as equal to 1 hat
	B_5.......	estimates 1 of money as equal to 1 hat
S_4............	....	estimates 1 of money as equal to 1 hat

This general demand curve may also be obtained directly from the conventional demand curve DD' and the conventional supply curve SS'. We may begin with the highest price. Only one hat is demanded or valued at $5.00. This is represented by the point D, whose co-ordinates are 1,5. When the price is $4.00, two hats are demanded and four hats are supplied. But the demand for two hats at $4.00 apiece *includes* the demand for one hat at $5.00; and the supply of four hats at $4.00 apiece *includes* the supply of three hats when the price is $3.00. Hence when the price is $4.00, one *additional* hat is supplied and one is demanded. That is to say, there are *two* hats which are valued at $4.00 apiece. This fact is represented by the points, 2,4 and 3,4. When the price is $3.00, three hats are demanded and three are supplied. But the demand for three hats at this price *includes* the demand for two hats when the price is $1.00 higher, and the supply of three hats at this price *includes* the supply of two hats when the price is $1.00 less. Hence when the price is $4.00 per hat, another *additional* hat is demanded and one more *additional* hat is supplied. In other words, there are *two* hats which are valued at the rate of $3.00 for 1 hat. This fact is represented by the points whose co-ordinates are 4,3 and 5,3 respectively. Proceeding in this manner we obtain the *general* demand curve $DP'D''$.

Having thus derived the general demand curve, how may we obtain the price at which the hats will be sold?

A glance at Figure 3 will show that the price is *not* given by the ordinate of the point at which this general demand curve $DP'D''$ crosses the supply curve SPS'. To determine the market price when we are given a general demand curve, all that we need to know is simply the total quantity available. We have seen that there are altogether four hats on the market. The ordinate of the curve $DP'D''$ corresponding to this supply is $P'4=3$. That is to say, the price will be $3.00 per hat, which agrees with the result obtained from DD' and SS'.

The general demand curve $DP'D''$ is subject to several limi-

tations, which must be kept in mind. (1) It does not tell us how many units exchanged hands. (2) It does not tell us whether the supply curve which it includes has a positive or negative slope, and, consequently, whether the price $P'4$ is stable or unstable. (3) It is based on the assumption of a fixed supply. This necessitates drawing up a new demand schedule whenever the total supply is changed. Suppose, for example, that in the foregoing hypothetical illustration, one new seller, S_5, enters the market, and is willing to sell one hat for $6.00. It is clear from the intersection of the ordinary supply curve SS' and the demand curve DD' that the new offer would not change the equilibrium price $P3=$ $3.00. The only effect that the introduction of the new offer would have on the ordinary supply curve SS' is to prolong it to the point 6, 6. However, its effect on the general demand curve $DP'D''$ would be a radical change, a shift to the right. In its present form the general demand curve tells us only what price corresponds to the fixed supply of four hats.

The foregoing discussion has an important bearing on problems connected with the derivation and interpretation of statistical laws of demand. All concrete laws of demand are based on observations, i.e., on statistical records, of prices paid and amounts sold or produced at various times. These records are, of course, nothing more than co-ordinates of various points on an unknown demand curve, affected by many disturbing factors. The statistical problem is to derive the most probable demand curve suggested by these observations. Whether, therefore, any statistical law of demand represents the demand function of both buyers and sellers, or that of buyers only, depends upon the nature of the statistical data on which it is based. If, for example, we are given the annual production of potatoes *including the amount left to rot in the ground, the amount fed to hogs, etc.*, and the corresponding prices, we have a series of points or observations, such as P' and D'' (Fig. 3), affected of course by disturbing factors. The law of demand derived from these observations will be a general demand curve. Such a de-

mand curve must not be put in juxtaposition with a supply curve. If, on the other hand, we are given the quantities sold and the corresponding prices, we have a series of observations or points such as P. The law of demand derived from these observations will be a particular (or buyers') demand curve. In practice, however, it may often be impossible to derive a purely buyers' or a purely general demand curve, owing to the unsatisfactory nature of the available statistics and to the fact that the two curves may coincide for part of the distance.

5. The most common demand curve is the locus of a point P whose abscissa represents the number of units demanded and whose ordinate represents the *price per unit* (see Fig. 2). There are, however, other kinds of demand curves. Thus, if in the above demand curve we let the ordinate represent *not* the price per unit, but the *total amount of money* (or any other specified article) given in exchange for the quantity represented on the X-axis, we have the integral curve which is frequently used for representing demand in international trade. In this study, however, only the more common demand curve will be dealt with.

IV. LIMITATIONS OF THE NEO-CLASSICAL LAW OF DEMAND

The limitations of the static law of demand have been discussed by Marshall[23] and, more especially, by Edgeworth.[24] At this point only two difficulties need be mentioned.

1) In *theory* the law of demand for any one commodity is given only on the assumption that the prices of all other commodities remain constant (the old *ceteris paribus* assumption). This postulate fails in the case of commodities for which substitutes are available. Thus when the price of beef is changed markedly, the prices of such rival commodities as mutton, veal, and pork cannot be supposed to remain constant. Likewise, the price of sugar cannot be increased beyond a certain point without affecting the prices of glucose, corn sugar, and honey.

[23] *Op. cit.*, pp. 109–19.

[24] Article on "Demand Curves," in Palgrave's *Dictionary of Political Economy*, I, p. 554.

2) The validity of the theoretical law is limited to a point in time. But in order to derive *concrete, statistical* laws our observations must be numerous; and in order to obtain the requisite number of observations, data covering a considerable period must be used. During the interval, however, important dynamic changes take place in the condition of the market. For a commodity like sugar, the principal dynamic changes that need be considered are the changes in our sugar-consuming habits,[25] fluctuations in the purchasing power of money, and the increase of population.

These (and other) difficulties led Professor Edgeworth to say ". . . . it may be doubted whether Jevons' hope of constructing demand curves by statistics is capable of realization."[26] It remained, however, for Professor Moore to show that these limitations may be overcome, and that concrete statistical demand functions may be derived which are exceedingly useful for a good many purposes.[27] Professor Moore overcomes the first of these limitations by substituting the more general law of demand of the mathematical school which is connected with the names of Walras and Pareto for the neo-classical variety; and the second, by utilizing the more modern tools of statistical analysis.

V. THE LAW OF DEMAND OF THE MATHEMATICAL SCHOOL

According to the mathematical school of economics, there is only one theoretically correct method of formulating the law of

[25] The per capita consumption of sugar in the United States increased from 52.8 pounds in 1890 to 107.5 in 1925.

[26] *Op. cit.*, p. 544.

[27] Henry Ludwell Moore, *Economic Cycles* (1914), chap. iv, "The Law of Demand"; *Forecasting the Yield and Price of Cotton* (1917), pp. 101–15, and chap. v, "The Law of Demand for Cotton"; "Empirical Laws of Demand and Supply and the Flexibility of Prices," *Political Science Quarterly*, December, 1919; "Elasticity of Demand and Flexibility of Prices," *Journal of the American Statistical Association*, March, 1922; "A Moving Equilibrium of Demand and Supply," *Quarterly Journal of Economics*, May, 1925; "Partial Elasticity of Demand," *ibid.*, May, 1926; "A Theory of Economic Oscillations," *ibid.*, November, 1926.

demand. It is to write the demand for any commodity as a function of *all* the prices in the economy under consideration; or—what amounts to the same thing—to write the price of any commodity as a function of the quantity demanded and of all the other prices.[28]

In symbolic form, this law of demand is

$$x_0 = f(x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_n) \tag{2}$$

where x_0 and x_1 are, respectively, the price and the quantity of the commodity in question, and $x_2, x_3, \ldots, x_n$ are the prices of all other commodities.

It is only through such a general formulation that all prices may be determined by a system of simultaneous equations and that an insight may be had into the tremendous complexity of our price economy. It is only through such a general formulation that the conditions may be mathematically deduced which give rise to the positively sloping demand curve.[29] Once the equation connecting the quantity demanded and all the prices has been determined, it is quite easy to deduce the relation between the quantity and its price on the assumption that all the other prices are kept constant, and thus obtain the neo-classical demand curve as a special case. This is accomplished by the mathematical operation of assigning constant values to all the variables in the equation except the two under consideration. And it may

[28] On the difference between the neo-classical and the mathematical statement of the law of demand, see Vilfredo Pareto, "Économie Mathématique" in *Encyclopédie des Sciences Mathématique,* Tome I, Vol. IV, Fascicule 4, pp. 593–94, 614–21, and especially 628–30; *Manuel d'Économie Politique,* pp. 579 ff. Leon Walras, *Eléments d'Economie Politique Pure* (4th ed., 1900), pp. 122–24, 208–16. Gustav Cassel, *The Theory of Social Economy,* chap. iv: "The Mechanism of Pricing." Henry L. Moore, "A Theory of Economic Oscillations," *Quarterly Journal of Economics,* November, 1926. For a criticism of Pareto's *Manuel* which has a bearing on his treatment of the law of demand, see Knut Wicksell's "Vilfredo Paretos Manuel d'économie politique," *Zeitschrift fur Volkwirtschaft Sozialpolitik und Verwaltung,* XXII (1913), 132–51, and especially p. 138.

The relation between the distribution of income and the law of demand is discussed by Pareto in "La legge della domanda," *Giornale degle Economisti,* X (January, 1895), 59–68.

[29] See references given in note 2, p. 5.

well be that the demand curve thus obtained will depend upon the magnitude of the constants which are assigned to the other variables.[30] Thus the demand curve for wheat when the price of rye is kept constant at $0.75 a bushel may be considerably different from what it is when the price is kept constant at $1.50 a bushel.

To derive the ordinary (neo-classical) demand curve as a special case of the general demand function of the mathematical school by assigning constant values to all the variables in the general demand function except the price and the quantity under consideration, is to effect an improvement over the neo-classical approach. The neo-classical economists simply *ignored* the other variables, without troubling themselves first to introduce them into their demand equation and then to assign to them constant values. They never faced the problem of the level at which each of the "other things" must be kept constant. The *ceteris paribus* postulate of classical fame must not, therefore, be confused with the method of mathematical ignoration described before.

Though the theoretical advantages of the Walras-Pareto over the neo-classical type of demand function are quite apparent, it may be asked whether its very generality renders it useless for practical purposes. If, in accordance with the general theory of equilibrium, the demand for any commodity must be written as a function of the prices of all the commodities and services in the economy under consideration, and if in the given economy there are as few as 100 commodities, and services, this means that the equation of demand for each commodity is a function of 100 variables. How can we deal with such complicated functions in any practical problem? The answer is that, although *in theory* it is necessary to deal with the demand function in all its complexity in order to show that the price problem is soluble,[31] *in*

[30] This will be true, for example, when the general demand function (see equations [*3b*] and [*3c*], p. 30) contains product terms in two or more of the variables.

[31] See Henry Schultz, "Mathematical Economics and the Quantitative Method," *Journal of Political Economy*, XXXV, No. 5 (October, 1927), pp. 702–6.

practice only a small advantage is gained by considering more than the first few highly associated variables.[32] Which are the most highly associated variables cannot be answered a priori. We either have recourse to experience, or else we proceed to select variables by trial and error, relying on the method of correlation to tell us how much of the phenomenon under consideration (dependent variable) is accounted for by each of the variables selected.

VI. THE STATISTICAL APPROACH TO THE PROBLEM

Although the mathematical law of demand is more general than the neo-classical variety, the two laws have this in common: they are both static laws; they relate to a point in time. Dynamic changes in demand that take place from time to time are not taken into consideration by them. But in order to bring the theory of consumption into closer agreement with the facts of our experience it is necessary to show how the demand curve *moves*—how it changes its position from time to time.[33] The statistical law of demand, as will be shown later,[34] may be made to show this; for it expresses the price of any commodity as a function, not only of the quantity demanded and the price of all other commodities, but also of *time.*

That the statistical law of demand must take into consideration the changes that the variables undergo in time is evident

[32] "If *A* in part determines *B*, when we disregard other factors, and *C* in part determines *B*, when we disregard all else, and similarly *D* and *E*, it is argued that all these part-determinations can be added together and the sum will finally determine *B*. But the error made lies in the supposition that *A*, *C*, *D*, *E*, etc., are themselves *independent*. In the universe as we know it all these factors are themselves to a greater or less extent associated or correlated, and in actual experience but little effect is produced in lessening the variability of *B* by introducing additional factors after we have taken the first few most highly associated phenomena." Karl Pearson, *Grammar of Science* (3d ed.), p. 172. Quoted by Henry L. Moore in his *Forecasting the Yield and Price of Cotton,* p. 162.

[33] Also changes in the *shape* of the "curve" must of course be measured. This may be accomplished by dividing the period of the observations into several convenient parts and determining the demand function for each part separately

[34] See pp. 40–42; 61–62; 74–75; 85–87.

from the very nature of the statistical approach. The determination of a statistical law of demand calls for numerous observations; and in order to obtain the requisite number of observations, data covering a considerable period must, as a rule, be used. During the interval, however, important dynamic changes take place in the market. The statistical approach must therefore deal with variables which are functions of time.

In mathematical symbols, the statistical law of demand is

$$x_0 = F(x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots\ldots, x_n, t) \qquad (3)$$

where the x's have the same meaning as in equation (2), and where t stands for time. If, in this dynamic law of demand, we give t a fixed value t_0, representing the particular date (or small interval of time) in which we happen to be interested, we obtain the Walrasian statical law of demand (2) as a special case. If we also give constant values to all the x's except x_0 and x_1, we obtain the neo-classical statical law of demand as another special case. In any inductive investigation, however, the statical laws of demand may only be approached, but never realized.

In most statistical investigations in demand it has been found convenient not to introduce time as an explicit variable into the demand equation, but to make adjustments in the other variables for such changes as are presumably due to time. The practice of reducing the quantities consumed or produced to a per capita basis and the money prices to a constant purchasing power basis; the use of first (or higher order) differences, link relatives, or trend ratios, etc., instead of the original observations are examples in point. When the data are so adjusted the dynamic, statistical law of demand becomes

$$X_0 = \phi(X_1, X_2, X_3, \ldots\ldots\ldots\ldots\ldots\ldots, X_n) \qquad (3a)$$

where the X's are the x's of (3), adjusted for time changes. Thus, if the adjustment is by the method of trend ratios, to be explained later, $X_i = \dfrac{x_i}{f_i(t_i)}$.

From the neo-classical point of view, all variables in the demand equation except the price and the quantity of the commodity under consideration may be looked upon as "disturbing factors." Since it is often possible—at least to a first approximation—to eliminate the disturbing factors collectively as well as individually, an important question presents itself as soon as we attempt to derive a law of demand from statistics, namely: Shall the resultant of the disturbing factors which affect the price of the commodity and the resultant of the disturbing factors which affect the quantity of the commodity be eliminated by a single statistical device? Or shall each factor be accounted for separately? Thus, suppose that the price of sugar is really a function of four variables: the quantity of sugar demanded, the price of glucose, the price of corn sugar, and the price of honey; but that we are chiefly interested in the relation between the price of sugar and the quantity of it which is demanded. Shall we first adjust the statistics of prices and consumption for the resultant of changes in the other variables—assuming that such an adjustment is possible—and then derive the desired relationship by considering the price of sugar as a simple function of only one variable, the quantity demanded, or shall we actually express the price of sugar as a function of all the four variables?

There can be no doubt but that the second method of approach is theoretically the more desirable solution of the problem, for it not only yields what Professor Moore has termed a "dynamic law of demand in its complex form," but it also permits us to derive an approximation to the static law of demand at the same time. This will be evident if our problem be put into algebraic form: Suppose that, following Professor Moore's notation,[35] we let X_0 be the percentage change in the price of a commodity, say sugar, and X_1 the percentage change

[35] For a full and suggestive treatment of the various methods, the reader must be referred to Professor H. L. Moore's *Forecasting the Yield and Price of Cotton*, chaps. v–vi.

in the amount of the commodity that is demanded. Then the dynamic law of demand may be presented as follows:

$$X_0 = \phi(X_1, X_2, X_3, \ldots, X_n) \qquad (3a)$$

where $X_2, X_3, \ldots, X_n$ are percentage changes[36] in other factors, say, prices of glucose, prices of honey, etc. Neither the form of the function ϕ nor the interrelations of $X_1, X_2, X_3, \ldots, X_n$, are known. In any practical problem, however, we may find effectual means of overcoming these difficulties. We may, "with due precautions against spurious results," experiment with different types of function ϕ and of the interrelations of $X_1, X_2, X_3, \ldots, X_n$, and select those types which enable us to determine X_0 with the degree of accuracy sufficient for the problem in hand.

As a first approximation we choose the simplest possible function,

$$X_0 = \phi(X_1, X_2, X_3, \ldots, X_n) = a_0 + a_1 X_1 + a_2 X_2 + a_3 X_3 + \ldots + a_n X_n \,. \qquad (3b)$$

As both the form of the function ϕ and the interrelations of the $X_1, X_2, \ldots, X_n$, are assumed to be linear, the method of multiple correlation immediately suggests itself for determining the values of the constants $a_0, a_1, \ldots, a_n$ and the correlation between X_0 and the right-hand members of (3*b*). An excellent illustration of this method is afforded by Professor Moore's derivation of the law of demand for cotton. Indeed, it would seem as if the method of multiple correlation were expressly invented for the purpose of deriving dynamic laws of demand.

This simple linear function will probably give good results in the majority of cases, especially where a sufficiently large number of variables are considered. If it should fail to do so, that is, if the error involved in determining X_0 from the simple function should turn out to be too large for the problem under

[36] The reason why Professor Moore uses percentage changes will become apparent later.

consideration, we may take as a second approximation to our function ϕ some more general equation, as

$$
\begin{aligned}
X_0=\phi(X_1, X_2, X_3, \ldots, X_n)=a_{00} & \\
\left.\begin{aligned}
&+a_{01}X_1+b_{01}X_1^2+c_{01}X_1^3+\ldots \\
&+a_{02}X_2+b_{02}X_2^2+c_{02}X_2^3+\ldots \\
&+a_{03}X_3+b_{03}X_3^2+c_{03}X_3^3+\ldots \\
&+\ldots\ldots\ldots+
\end{aligned}\right\} & \qquad (3c)
\end{aligned}
$$

and determine the constants of this equation and the correlation between X_0 and the right-hand member of $(3c)$.

Both $(3b)$ and $(3c)$ are examples of a "dynamic law of demand in its complex form." These equations, however, also include an approximation to the *static* law of demand as a special case. Thus, if in these equations we single out X_1 (the percentage change in the quantity of sugar demanded) as the important variable in relation to X_0 (the percentage change in price) by putting the other variables $X_2, X_3, \ldots, X_n$ equal to zero, we have examples of approximations to the *static* law of demand. "These variables $X_2, X_3, \ldots, X_n$ must be equal to zero since they severally represent *percentage changes*, and the general hypothesis in mind when the static law of demand is formulated is that there shall be no changes in other economic factors."[37]

Though the method of multiple correlation is, as we have just seen, the best method for dealing with the problem of the disturbing factors, it will not be employed in that part of this study which relates to demand; for we have excellent reasons for believing that the really important disturbing factors which obscure the relation between the prices and the amount of the commodity are not the changes in the prices of glucose, corn sugar, and honey, but the growing popularity of sugar as an article of consumption, the increasing population, and the changes in the general price level. And the effects of these long-time changes

[37] H. L. Moore, *op. cit.*, p. 152.

may be eliminated by one of two simple statistical devices quite as well as by the method of multiple correlation. These statistical devices, first applied to such problems by Professor Moore, are: (1) the method of relative changes, and (2) the method of trend ratios.

The method of relative changes consists of finding the functional relationship, not between the absolute prices and absolute quantities, but between the *relative change* in the price of the commodity and the *relative change* in the quantity demanded.

> By taking the relative change in the amount of the commodity that is demanded, instead of the absolute quantities, the effects of increasing population are approximately eliminated; and by taking the relative change in the corresponding prices instead of the corresponding absolute prices, the errors due to a fluctuating general price level are partially removed. If the observations should cover the period of a major cycle of prices, and the commodity under investigation should be a staple commodity, ... the above method of deriving the demand curve will give an extremely accurate formula summarizing the relation between variations in price and variations in the amount of the commodity that is demanded.[38]

As a measure of relative change we may take either the *percentage change* in the value from one year to the next or the *ratio* of the given year's value to that of the preceding year (link relatives). In this study, however, we shall use only ratios or link relatives in preference to percentage changes, primarily because the former, unlike the latter, are always positive numbers. This is a great advantage where it is desirable to work with the logarithms of the figures.

The method of trend ratios derives the demand curve, not from the absolute prices and corresponding absolute quantities, but from the *ratios* of these prices and quantities to their respective trends. The *rationale* of this method may be based on the following considerations.

If, during the period when our observations were taken, "all other things" had remained equal as theory demands, we should have no secular trend either of prices or of quantities.

[38] H. L. Moore, *Economic Cycles*, pp. 69–70.

The existence of a secular trend in either series is *prima facie* evidence that "all other things" did not remain equal, that there was one or more disturbing factor or element. It is the disturbing elements which give rise to the trend and which create a different "normal" from time to time. Hence it follows that by taking the *ratio* of the actual (observed) prices to normal or trend prices, we eliminate, to a first approximation, the effect of the long-time disturbing elements on the price of the commodity under consideration. Likewise, by taking the *ratio* of the corresponding quantities to their trend, we eliminate approximately all the long-time disturbing factors influencing the supply. By taking the ratios of our variables to their respective trends we are practically overcoming the chief difficulties which, according to Edgeworth and others, lie in the way of deriving statistical laws of demand. For our data, though extending over a period of years, may, when thus adjusted, be conceived of as representing approximately observations taken at a given point in time—at least for practical purposes.

Though the method of trend ratios is, in general, to be preferred to the method of relative changes or link relatives, on the ground that it gives a more adequate solution of the problems which are faced in the derivation of the static law of demand, it would be erroneous to assume that it is always superior to the method of link relatives. *All of these statistical devices are to be valued according to their efficacy in enabling us to lay bare the true relationship between the phenomena under consideration.* An *ideal* method would eliminate entirely all of the disturbing factors. We should then obtain *perfect correlation* between changes in the quantity demanded and corresponding changes in price. That is to say, the observations would all lie on the demand curve—there would be no "scatter" of points around the curve. The efficacy of any *practical* method is to be valued according to the closeness with which it meets the ideal or a priori requirements. Judged by this standard, the method of link relatives not infrequently gives

better results than the method of trend ratios. At other times, especially when it is necessary to use data which are affected by abnormal conditions, it will be found desirable first to correct the price series by an index number of prices, and the quantity series by the census figures for population, before proceeding to apply any of the other methods.

In the next two chapters all of these methods, with the exception of the method of multiple correlation, will be applied to the same problem—the derivation of the law of demand for sugar—with the view of testing their relative advantages and disadvantages. More specifically, the law of demand for sugar in the United States will be derived: (*a*) by the method of link relatives (*b*) by the method of trend ratios. Both methods will be applied to the observed or unadjusted data for prices and consumption as well as to data which were first corrected for changes in the general level of prices and for population growth. Attention will also be called to the type of trend which is most suitable for the purpose of the problem in hand and to the type of equation which is most suitable for the purpose of deriving the best value of the coefficient of elasticity.

CHAPTER II

THE LAW OF DEMAND FOR SUGAR DERIVED FROM UNADJUSTED DATA

I. THE METHOD OF LINK RELATIVES APPLIED TO THE UNADJUSTED DATA

In Table I of Appendix II there are recorded, for the period from 1890 to 1914 inclusive, two unadjusted series: the total yearly consumption of sugar in the United States, and the average annual wholesale prices at New York of refined sugar in cents per pound.[1] The New York prices are taken because New York was the dominant sugar market during the period covered by this investigation; prices in all markets throughout the United States were fixed by the New York price. In the same table there are also recorded the link relatives of consumption and of prices (columns 4 and 5).

There is no correlation between the crude data for consumption and prices[2] (columns 2 and 3). On the other hand, there is considerable correlation between the link relatives of total consumption (X) and the link relatives of money prices (Y), the coefficient of correlation[3] being -0.68 ± 0.07. A graphic presentation of this inverse relationship is to be found in Figure 4.

I. THE PROBLEM OF FITTING THE DEMAND CURVE

When the link relatives are plotted on a scatter diagram (see Fig. 5), the nature of the correlation between the two

[1] The statistics were derived from Truman G. Palmer's loose-leaf service, *Concerning Sugar*, pp. E-54, *A, B, C* (consumption), and p. E-8 (prices), published by the United States Sugar Beet Association, Washington, D.C.

[2] $r=-0.06\pm0.14$.

[3] The means of the link relatives of consumption and the link relatives of prices are 1.0415 and 0.9936 respectively. The standard deviations are 0.062,925 and 0.096,696, respectively.

variables is more clearly exhibited. It will be seen that, considering the number of observations and their distribution, one would hardly be justified in assuming skew relationship. Hence the problem of deriving the law of demand for sugar reduces to the problem of deriving the equation of the "best-fitting" straight line.

The common method of fitting a straight line to data involves the arbitrary selection of one of the variables as the independent variable X and the assumption that an observed

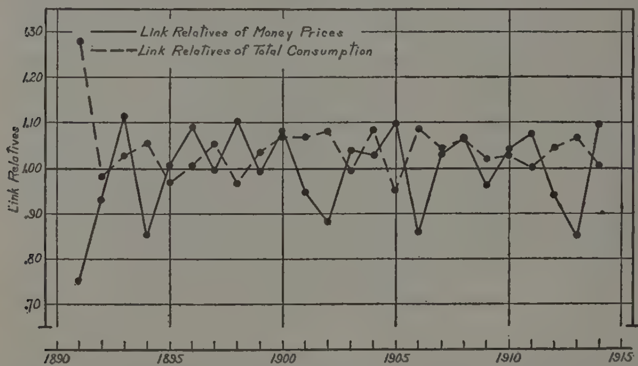

FIG. 4.—Inverse correlation between the link relatives of the total consumption of sugar in the United States and the link relatives of the wholesale money prices of sugar.

point fails to fall on the line because of an "error" or deviation in the dependent variable Y alone, the X variable being allowed no deviation. The result of this method is that we get one straight line if we treat one of the variables as the independent variable, and a quite different one if we treat the other variable as the independent variable. Thus, in Figure 5, the straight line A (the regression of Y on X) gives the most probable relative change in the price of sugar (Y) corresponding to a given relative change in consumption (X). The assumption made in deriving this line is that the statistics of consumption are free

from error, the failure of the points to fall on the line being due to errors in the price data only. The line A' (the regression

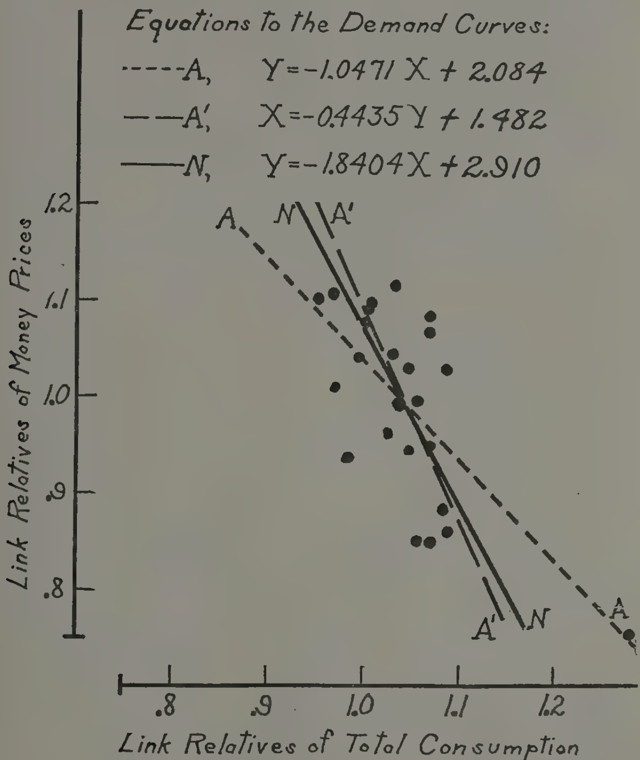

FIG. 5.—Scatter diagram of the link relatives of the total consumption of sugar and the corresponding link relatives of the wholesale money prices of sugar, with the lines of regression (A and A') and the "line of best fit" (N).

of X on Y) gives the most probable relative change in consumption corresponding to a given relative change in price. The assumption made in deriving this line is that the statistics of

prices are free from error, the failure of the points to fall on the line being due to errors in the statistics of consumption alone. In deriving line A the sum of the squares of the *vertical* distances of the points from the line was made a minimum. In deriving line A', on the other hand, the sum of the squares of the *horizontal* distances of the points from the line was made a minimum. *In fact, in fitting line A the observed values in X are regarded as having a weight infinitely greater than those in Y, while in fitting line A' the reverse assumption is made.*

It will be seen from Figure 5 that there is a considerable difference between the *slopes* of the two lines. That is to say, we get two quite different demand curves from the same equation according as X (consumption) or Y (prices) is assumed to be free from error. The difference is sometimes such that one curve shows that the demand is elastic, while the other shows that the demand is inelastic.

Now in practically all cases of observed data relating to demand and supply, the X variable is as subject to errors or deviations as the Y variable. We know, for example, that the statistics of sugar consumption are as subject to error as the statistics of sugar prices, if not more so. It therefore appears that a better-fitting straight line will be obtained by assuming that an observed point fails to fall on the line because there is an error in both X and Y. The curve-fitting method which is based on this assumption is much more complicated than the ordinary method. To the best of our knowledge, its *direct* application has thus far been confined to linear equations. No one has as yet succeeded in extending it (without making tedious empirical approximations) to the fitting of even such a comparatively simple non-linear curve as the second degree parabola. Even if the method is to be applied to linear equations only, the further assumption has to be made that the errors of X and Y have a *constant ratio*. When this ratio is equated to unity, that is to say *when the observations in X are given the same weight as those in Y*, or when the two variables are assumed to be equally

accurate or equally inaccurate, this method has a simple geometrical interpretation: it makes the sum of the squares of the *normals*—the perpendicular distances of the points from the curve—a minimum.[4] The line N (Fig. 5) was fitted on this assumption. It may, therefore, be looked upon as the "line of best fit" or the "line of mutual regression." Within the limits of observation, it is the best "demand curve" that may be derived from the data at hand. It gives the most probable relation between relative consumption changes and relative price changes for the interval 1890–1914.

2. THE LAW OF DEMAND IN TERMS OF LINK RELATIVES

The equation of line N is

$$Y = -1.8404\,X + 2.910 \tag{4}$$

the origin being at 0,0. This means that, based upon the experience from 1890 to 1914, an increase (or decrease) of one *point* (or, roughly, 1 per cent) in the link relative of consumption, is *on the average,* associated with a decrease (or increase) of 1.84 *points* in the relative price. Another way of saying the same thing is that an increase (or decrease) of one *point* in the relative price is, *on the average,* associated with a decrease (or increase) of $1/1.8404 = 0.54$ of a *point* in the relative consumption. If line A be taken as the demand curve, the corresponding figure is $1/1.0471 = 0.96$ of a *point*, and if line A' be taken as the demand curve, it is 0.44 of a *point*.

By means of this equation it is possible to estimate the probable change in the price corresponding to a given change in consumption, or the probable change in consumption corresponding to a given change in price. In 1913 the consumption of sugar in the United States was 4,192,000 short tons. (See

[4] The hypothesis that the independent variable is as subject to error as the dependent variable has been considered by Adcock (1877 and 1878), Kummell (1879), Merriman (1891), Pearson (1901), Reed (1921), Gini (1921), Uhler (1923 and 1925), Pietra (1924), Glauert (1925), Whipple (1926), Rhodes (1927), and Roos and Oppenheim (1928). See partial bibliography at the end of this chapter.

Table I of Appendix II.) In 1914 the consumption was 4,212,000 tons. What was the probable price of sugar in 1914? The change in consumption as measured by the link relative was $4{,}212{,}000 \div 4{,}192{,}000 = 1.005$. Substituting this value for X in the equation $Y = -1.8404\ X + 2.910$ and solving for Y, we find that the probable price would be 1.0604 of the price for the preceding year. Since the price for 1913 was 4.278 cents, this value of Y would give 4.536 cents as the probable price for 1914. The actual price was 4.683 cents. In like manner we can estimate the probable change in consumption corresponding to a given change in price.[5]

3. THE LAW OF DEMAND IN TERMS OF ABSOLUTE QUANTITIES

Equation (4) gives the most probable relation which existed between the relative changes in the consumption and the corre-

[5] This example illustrates the type of problem which arises most frequently in practical forecasting, namely, having estimated or observed the value of one variable, to find the probable value of the other. In the solution of this problem the given value is assumed to be free from errors of observation and the "line of best fit" is used like an ordinary line of regression. The solution would be different, however, if we were given the observed values of *both* variables for any one year and were required to compute the corresponding probable values, since errors in both variables would then have to be taken into account.

Thus, if the equation to the "line of best fit" be $Y = AX + B$, then the probable or adjusted pair of values X_1' and Y_1' corresponding to the given or observed pair X_1 and Y_1 would have to be obtained from the following relationships:

$$X_1' = X_1 - \frac{A}{G + A^2}(AX_1 + B - Y_1)$$

$$Y_1' = Y_1 + \frac{G}{G + A^2}(AX_1 + B - Y_1)$$

where G is the ratio of weight of the independent variable to that of the dependent variable. (See reference *b* in the bibliography.) In deriving equation (4) as well as all the other "lines of best fit" used in this study, G was taken as unity.

If in the above equations we substitute for X_1 and Y_1 their observed values for 1914 (see Table I of Appendix II) by putting $X_1 = 1.005$ and $Y_1 = 1.095$, we obtain as the most probable values for that year $X_1' = 0.9905$ and $Y_1' = 1.0871$. The latter are the co-ordinates of the point at which the normal through the observed point crosses line N (Fig. 5).

sponding relative changes in the price of sugar during the interval 1890–1914. It is the law of demand for sugar in the link-relative form. A simple transformation, however, makes it possible to take the next step—the derivation of the law of demand for any one year in terms of absolute quantities. By definition, $X=\frac{x}{x_0}$ and $Y=\frac{y}{y_0}$, where x_0 and y_0 are respectively the consumption and prices in the preceding or base year. By making these substitutions in (4), the law of demand in the absolute form becomes

$$y/y_0=-1.8404\ x/x_0+2.910$$

$$\text{or } y=-1.8404(y_0/x_0)x+2.910y_0 \tag{5}$$

which may also be written[6] as

$$x=-0.5433\ (x_0/y_0)y+1.581\ x_0\ . \tag{5a}$$

If, for example, it is desired to have the law of demand for sugar in 1914, all that we have to do is to find the observed consumption (x_0) and the observed price (y_0) for the "previous year," or 1913, and to substitute these values in either (5) or (5*a*). The observed consumption and price for each year between 1890 and 1914 are shown in Table I of Appendix II. It will be seen that in 1913 the consumption of sugar amounted to 4,192,000 tons (x_0), and the price was 4.278 cents (y_0). Substituting these values in (5) we obtain

$$y=-0.000{,}001{,}878\ x+12.449\ .$$

This form is very convenient for estimating price from consumption. Thus, the total consumption of sugar in 1914 amounted to 4,212,000 tons. What was the probable price? Substituting this value of x in the foregoing equation we obtain

[6] It must be kept in mind that equation (5*a*) is not the "regression of x on y" in the sense in which this expression is used in statistical literature. It is simply another form of (5).

$y=4.539$, or the probable price for 1914 would be 4.539 cents. The observed price was 4.683 cents—a difference of 3 per cent.

If, on the other hand, it were desired to estimate consumption from price, it would be more convenient to substitute x_0 and y_0 in equation (5a), which, for 1914, becomes

$$x=-532{,}000\ y+6{,}628{,}000\ .$$

By way of comparison it is interesting to have the corresponding equations for 1904 and 1894. These are

$$x=-335{,}000\ y+4{,}515{,}000$$

$$\text{and } x=-240{,}000\ y+3{,}377{,}000\ .$$

That is to say, a rise or fall in the price of sugar of one cent per pound would have decreased or increased consumption by 240,000 tons in 1894, by 335,000 tons in 1904, and by 532,000 tons in 1914. Corresponding equations may be derived for all the other years between 1890 and 1914. *Such equations afford a measure of the shifting of the demand curve from year to year as a result of dynamic changes.* In a subsequent section, however, a more satisfactory method of determining these equations will be presented.

4. THE STANDARD ERROR OF THE DEMAND CURVE

It is desirable to have a measure of the accuracy or "goodness of fit" of line N and to compare it with that of line A and line A'. The constant that is most commonly used to measure the "goodness of fit" of a curve is the standard error, or the root-mean-square deviation of the actual observations about the fitted curve. The standard error enables us to tell the *degree* of scatter, or the limits within which any proportion of the observations (points) are distributed about the line or curve under consideration. Assuming that the points are symmetrically distributed about the line or curve, we know from the Table of the Probability Integral that 68 per cent of all observations fall within $\pm$ the standard error; 95 per cent between

$\pm$ twice the standard error; and 99.7 per cent between $\pm$ three times the standard error. All other things—the number of constants in the equations, the nature of the underlying assumptions, etc—being the same, the "goodness of fit" of a curve is inversely proportional to its standard error. That curve gives the best fit which has the smallest standard error.

In the case of the three lines of Figure 5, however, we should not be justified in comparing standard errors for the purpose of selecting the line of best fit, because the standard errors of these lines are not really comparable. The errors in question are: $S_A = 0.0708$, $S_{A'} = 0.0461$, $S_N = 0.0413$, where S_A, $S_{A'}$, and S_N denote the standard errors of the lines A, A', and N respectively.[7] That is to say, 68 per cent or, say, two-thirds of all the points or observations lie within a distance of ± 0.0708 units from line A, of $\pm$ 0.0461 units from line A', and of only ± 0.0413 units from line N. *But these distances are not measured in the same direction.* Thus, S_A represents a *vertical* distance of ± 0.0708 units from line A (the regression of Y on X), $S_{A'}$ represents a *horizontal* distance of ± 0.0461 units from line A' (the regression of X on Y), while S_N represents a *perpendicular* distance of ± 0.0413 units from line N. In other words, the standard errors are measured in *different units*. Thus S_A is measured in terms of Y (*prices*), $S_{A'}$ in terms of X (*consumption*), and S_N in terms of both X and Y.[8] These errors are, therefore, not comparable.

[7] These errors were computed from the following formulas:

$$S_A = \sigma_Y(1-r^2)^{\frac{1}{2}}$$

$$S_{A'} = \sigma_X(1-r^2)^{\frac{1}{2}}$$

$$S_N = \left[\tfrac{1}{2}(\sigma_X^2+\sigma_Y^2) - \tfrac{1}{2}\sqrt{(\sigma_X^2+\sigma_Y^2)^2+4r^2\sigma_X^2\sigma_Y^2}\right]^{\frac{1}{2}}$$

where σ_Y is the standard deviation of the variable Y, σ_X is the standard deviation of the variable X, and r is the coefficient of correlation between the variables. The first two formulas are well known. The third is given by Pearson, *loc. cit.*

[8] This follows from the fundamental assumption made in fitting line N, namely, that an observed point fails to fall on the line because of *two* errors or deviations: a (vertical) deviation in the dependent variable Y, and a (horizon-

To obtain comparable standard errors, the deviations about each of the three lines should be measured *in the same direction.* That is to say, the three sets of deviations should all be measured either in the vertical direction, or in the horizontal direction, or in the direction normal to the line under consideration. But if we do this, we shall get different results, depending upon the direction selected. Thus, if the deviations about the three lines be measured in the vertical direction, we know that line A will have the smallest standard error, for it was fitted by making the sum of the squares of the vertical deviations a minimum. If the deviations about the three lines be measured in the horizontal direction, we know that line A' will have the smallest standard error, for it was fitted by making the sum of the squares of the horizontal deviations a minimum. If the deviations about the three lines be measured in directions perpendicular to these lines, we know that line N will have the smallest standard error, for it was fitted by making the sum of the squares of the normal deviations a minimum.

We conclude, therefore, that while the standard error of line N ($S_N = 0.0413$) has a perfectly definite meaning, it cannot be compared with the standard errors of line A and line A'. The three lines are the results of three different assumptions regarding the distribution of errors in X and Y. The justification of selecting line N as "the line of best fit" must, therefore, be based

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tal) deviation in the independent variable X. The normal deviation is simply the resultant of these two deviations, the three quantities being related in the same way as the hypotenuse and the two sides of a right triangle.

Let X_1 and Y_1, and X_2 and Y_2, etc., be the co-ordinates of the observed points. Let X_1' and Y_1', X_2' and Y_2', etc., be the co-ordinates of the corresponding adjusted points, i.e., the points at which the normals intersect the "line of best fit." Let W be the number of points or of pairs of observations. The vertical deviations are $Y_1 - Y_1'$, $Y_2 - Y_2'$, etc. The corresponding horizontal deviations are $X_1 - X_1'$, $X_2 - X_2'$, etc. Hence the sum of the squares of the normal deviations may be written as

$$WS_N = \Sigma(X - X')^2 + \Sigma(Y - Y')^2 .$$

It is this function which was made a minimum in the fitting of line N.

on the fundamental considerations which were adduced, and not on the size of its standard error.

5. COEFFICIENT OF THE ELASTICITY OF DEMAND

Having determined the law of demand for sugar as well as its standard error, and having indicated the great care that must be exercised in comparing standard errors, we are in a position to compute the coefficient of the elasticity of demand.

It will be recalled that by definition the coefficient of the elasticity of demand is

$$\eta = \frac{dx}{x} \Big/ \frac{dy}{y} = \frac{dx}{dy} \cdot \frac{y}{x} .$$

As the law of demand for any one year is, as we have seen, given by (5), all that we have to do in order to derive the coefficient of the elasticity of demand for any one year is to find $\frac{dx}{dy}$ from (5) and to multiply the result by y/x. We have

$$\frac{dx}{dy} = 1 \div \frac{dy}{dx} = -\frac{1}{1.8404} \cdot \frac{x_0}{y_0} = -0.5433 \frac{x_0}{y_0}$$

$$\therefore \eta = \frac{dx}{dy} \cdot \frac{y}{x} = -0.5433 \frac{y}{y_0} \cdot \frac{x_0}{x} . \tag{6}$$

But $\frac{y}{y_0} = Y$ and $\frac{x_0}{x} = \frac{1}{X}$. Hence (6) may be written as

$$\eta = -0.5433\, Y/X . \tag{6a}$$

This result, however, might have been obtained directly from (4) by finding

$$\frac{dX}{dY} \cdot \frac{Y}{X} = \eta = -0.5433\, Y/X .$$

From (6*a*) we see that the coefficient of the elasticity of demand varies from point to point on the demand curve N (Fig. 5). It is higher for high prices (or low consumption) than it is

for low prices (or high consumption). (See, however, note 20, p. 153.) Its satisfactory representation, therefore, calls for a graph which would depict it as a continuous function either of consumption or of price. Such a graph will be presented later in connection with a more satisfactory method of deriving the demand curve for sugar. For the time being we shall content ourselves with determining the coefficient at only three points on the demand curve (4).

When $X=1$, that is to say, when there is no change in the consumption of sugar between any two consecutive years, then $\eta=-0.5433\ (1.0696)=-0.58$, or a reduction in the price of sugar of 1 per cent will increase consumption by 0.58 of 1 per cent. When $X=1.1$, that is to say, when "this year's" consumption is already 10 per cent above that for last year, then $\eta=-0.5433\ (0.8856/1.1)=-0.44$, or a reduction of 1 per cent in price will then increase consumption by only 0.44 of 1 per cent. On the other hand, when $X=0.9$, that is to say, when "this year's" consumption is only 90 per cent of that for the preceding year, then $\eta=-0.5433\ (1.2536/0.90)=-0.76$, or a decrease in the price of 1 per cent will, under those conditions, increase consumption by as much as 0.76 of 1 per cent. Had we chosen line A or line A' for our demand curve, we should have obtained considerably different values for our coefficients. It is evident, therefore, that the methods of fitting the demand curve have a great influence on the value of the coefficient of elasticity of demand.

II. THE METHOD OF TREND RATIOS APPLIED TO THE UNADJUSTED DATA

Thus far we have derived the law of demand for sugar from unadjusted data by the method of link relatives. We have made one *general* correction for all the disturbing factors affecting price and another *general* correction for all the disturbing factors affecting consumption by correlating the link relatives of the two variables rather than their absolute magnitudes. But,

as we have pointed out in chapter i, there is another method which may be used for making these general corrections—the method of trend ratios. What problems does this method raise which have not already been considered? Does it yield more satisfactory results than those which have just been obtained? We are now ready to address ourselves to these questions.

The method of trend ratios, it will be recalled, derives the demand curve, not from absolute prices and corresponding absolute quantities, but from the *ratios* of these variables to their respective trends (or "normal" values). The *rationale* of this method has already been explained[9] and, therefore, need not be repeated here. Suffice it to say that this method postulates a knowledge of the "normal" consumption and the "normal" price for each year, the assumption being that the "normal" values are slowly, smoothly changing quantities about which the observed quantities fluctuate. And as we have no a priori knowledge of what constitutes a "normal" consumption or a "normal" price for any single date, these norms must be obtained by fitting good empirical curves to the data before us.[10]

I. THE PROBLEM OF SELECTING A GOOD TREND

Now in the derivation of a good empirical trend, the statistician is faced with a fundamental difficulty: he must choose a curve which reproduces the underlying movement of the data without bending or twisting itself so as to conform to the extreme sinuosities, and which at the same time gives a good "fit" as judged by some arbitrary criterion—say, the mean square error. These requirements, however, are mutually destructive, and the statistician must exercise his judgment, in view of all the known facts, in effecting a happy compromise between these requirements. The whole question has been well stated by E. C. Rhodes:

[9] Cf. pp. 32–34.

[10] Sometimes it is also advisable to have recourse to the graphic method and to the method of moving averages.

Given a series of points which require to be smoothed, we have always before us two principles, (1) that we must obtain a curve without the sinuosities which are not inherent in the data, and (2) we must get a curve which "fits" by least squares. These two principles are mutually inconsistent. On the one hand we can obtain the curve which actually goes through the points—here the sum of the squares of the errors is zero, but the curve is composed of perturbations or sinuosities. On the other hand we can obtain a perfectly smooth curve, a straight line, a parabola, a sine curve, or whatever may be chosen to suit the given series, and we obtain a large mean square error.

Somehow by careful balancing of these two mutually destructive principles we can obtain for any data a curve which is *reasonably* smooth and which at the same time gives a series of errors which have a *reasonably* small mean square error. The smoothness we obtain is not the best, nor is the sum of the squares of the errors a minimum.[11]

For most statistical series, it is impossible to get an objective test of the wisdom of one's compromise between smoothness and goodness of fit. With the data before us, however, we may get at least a rough check on the suitability of the "compromise curve" for the problem in hand.[12] The main, if not the only, object in fitting trends to consumption and prices is to enable us to remove the disturbing factors which affect these variables and mask or hide their true relationship; for we know that when all of the disturbing factors have been eliminated, the correlation between the variables is perfect. Hence those trends are most suitable which best accomplish this object. That is to say, of all the pairs of trends that may be obtained by selecting one trend for consumption and one for prices, that pair should be selected whose deviations (trend ratios), other things being equal, are most highly correlated.[13] If the curves which are indicated by this test are the same, or nearly the same, as those which would have been selected anyway on account of their appearing to be

[11] *Tracts for Computers, No. VI,* "Smoothing," pp. 43–44 (Cambridge University Press, 1921).

[12] See pp. 33–34, and Table IV, p. 53.

[13] Assuming of course, that due precautions are taken against spurious correlation.

the best compromise between smoothness and goodness of fit, the confidence which statisticians are wont to place in the latter type of curve is not misplaced.

In Figure 6 there are graphed, for the period from 1890 to 1914, inclusive, the consumption of sugar in the United States, and four computed trends[14]—a straight line, a quadratic, a cubic, and a quartic (biquadratic). It is clear from the graphs

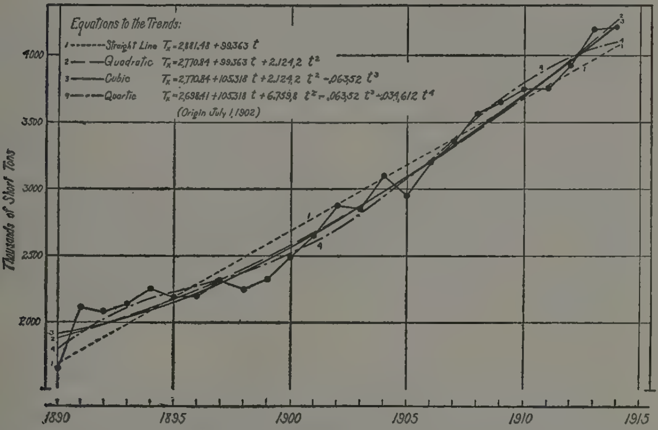

FIG. 6.—Total consumption of sugar in the United States, 1890–1914, and four fitted trends.

that neither the straight line nor the quartic is a satisfactory trend. The former is not "flexible" or "sensitive" enough; the latter is too flexible to yield satisfactory estimates of "normal" consumption. The parabola and the cubic, on the other hand, give excellent trends, and they are so nearly alike that it is impossible to tell which is the better of the two. As a choice has to be made, we take the cubic as giving the "best" trend of consumption. It is shown separately in Figure 7.

[14] In the equations to the trends, x = actual or observed consumption; T_x = computed consumption; t = time in years, the origin being at July 1, 1902.

In Figure 8 are graphed the prices and six computed trends[15] —a straight line, a quadratic, a cubic, a quartic, a quintic, and a curve derived by Rhodes' method of smoothing[16] and designated by the letter R. The latter consists of arcs of high-order parabolas, each of which runs into its neighbors and has a contact of the first order only. Rhodes provides a rough method of deciding the number of points that should be employed in smoothing

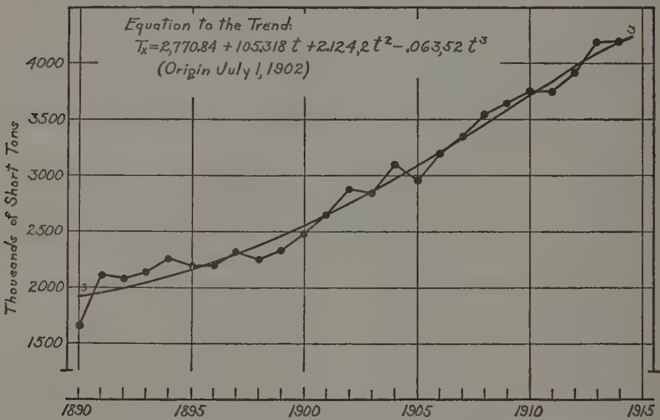

FIG. 7.—The "best" trend of the consumption of sugar in the United States, 1890–1914.

by his method. This test suggests a fifteen-point smoothing. A fourth-order parabola was fitted to the first fifteen points (i.e., to the prices from 1890 to 1904, inclusive) by the method of least squares, and the arc of this quartic between 1897 and 1898 was used as part of the final smoothed curve R. That is to say, this first quartic fixed the smoothed or "normal" prices for 1897 and 1898. A parabola of the same order was then fitted to the

[15] In the equations to the trends, y=actual or observed price; T_y=computed price; t=time in years, the origin being at July 1, 1902.

[16] E. C. Rhodes, *op. cit.*

next fifteen points (i.e., to the prices from 1891 to 1905, inclusive) in such a way that it joined on to the arc of the first curve at the smoothed ordinate for 1898 and had the same tangent at this point as had the first curve. The arc of this (second) curve between 1898 and 1899 was then used as part of our final curve. Proceeding in this way we derived the *R* curve between 1897

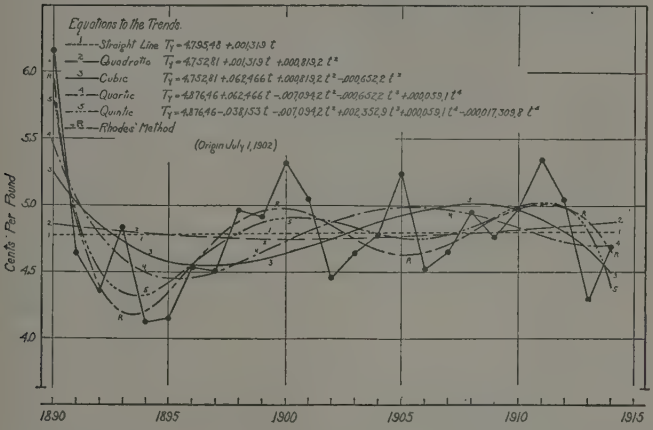

FIG. 8.—Wholesale money prices of refined sugar at New York, 1890–1914, and six fitted trends.

and 1908. The *tails* of the curve (i.e., the segment from 1890 to 1897 and the segment from 1909 to 1914) were obtained by using more of the first quartic that was found than its middle arc, and more of the last quartic that was found than its middle arc.

It is clear from Figure 8 that the straight line and the second-degree parabola are thoroughly unsatisfactory as trends. They are too "rigid" or "inflexible" and cannot, therefore, conform to the underlying movement of the observed data. The Rhodes curve and the quintic—especially the former—are, on the other hand, too "flexible" or "sensitive"; their fit is "too

good." Most statisticians, we dare say, would, by inspection, choose either the cubic or the quartic as being a happy compromise between smoothness and goodness of fit, the chances being slightly in favor of the cubic, as it is the simpler of the two curves. Of course, if the selection were to be guided by no other consideration than that of "goodness of fit," the palm would

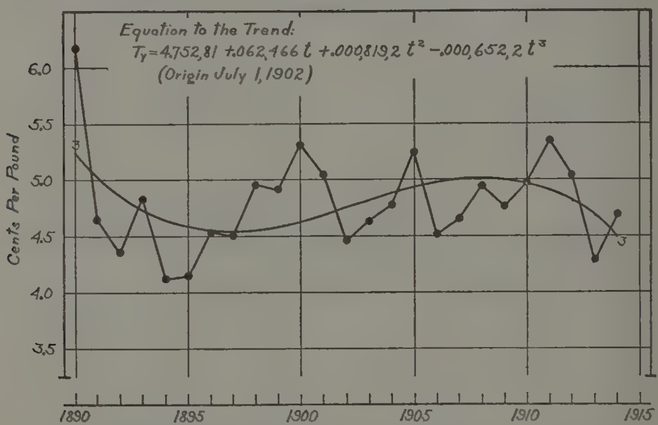

FIG. 9.—The "best" trend of wholesale money prices of refined sugar at New York, 1890–1914.

have to be awarded to the Rhodes curve.[17] The degree to which the cubic provides a satisfactory description of the underlying movement of sugar prices from 1890 to 1914 is best shown by Figure 9.

We are now ready to apply our test; that is to say, we are ready to determine which set of trend ratios is most highly correlated. Now the full application of this test would require the

[17] The sum of the squares of the residuals of each of the six curves is shown in the following table:

Curve	Σv^2	Curve	Σv^2
Straight line	4.681	Quartic	3.149
Quadratic	4.630	Quintic	1.959
Cubic	3.681	Rhodes curve	1.890

computation of twenty-four correlation coefficients[15]—by no means an easy task. We shall, however, effect a great saving of time if we limit our computations only to those series which appear, by inspection, to yield promising results. A comparison of the curves in Figure 6 with those in Figure 8 suggests that no useful purpose will be served by including in our computations the series of ratios derived from curves 1, 2, and 4 of Figure 6,

TABLE IV

SUMMARY OF THE CORRELATION BETWEEN THE TREND RATIOS OF THE TOTAL CONSUMPTION OF SUGAR (X) AND THE CORRESPONDING TREND RATIOS OF SUGAR PRICES (Y)*

Constants	Correlation between X_3 and			
	$Y_{[illegible]}$	$Y_{[illegible]}$	$Y_{[illegible]}$	$Y_{[illegible]}$
Coefficient of correlation	[illegible]	[illegible]	[illegible]	[illegible]
Coefficient of elasticity of demand (at the mean)†				
1. Regression of X on Y	[illegible]	[illegible]	[illegible]	[illegible]
2. Regression of Y on X	[illegible]	−0.440	−0.386	−0.242
3. "Line of best fit"	[illegible]	−0.530	−0.613	−0.474

* X denotes the ratio of the observed consumption to the trend of consumption, and Y denotes the ratio of the observed price to the trend of prices. The subscripts denote the curve which was used as the trend. Thus X_3 means that in computing the series of consumption ratios curve 3 (Fig. 6) was used [illegible]

† The coefficients of elasticity relate to the point $X=1$, $Y=1$.

and from curves 1 and 2 of Figure 8. The exclusion of these curves leaves only five series of ratios to be considered—one series of consumption ratios and four series of price ratios—and only four coefficients to be computed. The results of the computations are presented in Table IV.

From this table three conclusions may be drawn:

[15] We fitted four curves to consumption and six curves to prices. As each curve yields a series of trend ratios, there are 4 × 6 = 24 possible combinations between the series of consumption ratios (X) and the series of price ratios (Y).

1) The use of comparatively simple curves for the purpose of obtaining "normal" consumption and "normal" prices yields better (a priori) results than those obtained through the use of more complex curves. The highest correlation ($r = -0.78$) between the trend ratios is obtained when a cubic is taken as the trend of prices as well as of consumption. The degree of correlation diminishes as the complexity or sensitiveness of the price trend is increased until, as in the case when the Rhodes curve is used, the very existence of a correlation becomes somewhat doubtful.

2) The lower the correlation between X and Y, the greater the divergence between the coefficients of elasticity derived from the ordinary lines of regression. The common practice of arbitrarily selecting one of the variables as the independent variable, without considering the effect of the selection on the elasticity of demand, is apt to lead to erroneous conclusions.

3) Even if we choose the *worst* trends, thereby obtaining a low correlation between the resulting series of trend ratios (see column 5), recourse to the "line of best fit" still yields a fair approximation to the best value of the coefficient of elasticity of demand. (Compare column Y_R with column Y_3.) In fact, the lower the correlation between the variables, the more necessary it is to use the line or curve which makes the sum of the squares of the normals a minimum.

2. THE PROBLEM OF SELECTING A GOOD DEMAND CURVE

Having thus determined the trends of consumption and prices which are most satisfactory for our purposes, we are in a position to consider somewhat more fully than we have had occasion to do heretofore the further steps in the statistical determination of the law of demand by the method of trend ratios.

In Table II of Appendix II there are recorded, for the period from 1890 to 1914, the observed consumption and the observed prices of sugar; the "normal" consumption and the "normal" prices as determined from the most satisfactory trends; and the consumption ratios (X) and the price ratios (Y). The con-

sumption ratios and price ratios are highly correlated, the coefficient of correlation being, as we have already pointed out, —0.78. A graphic presentation of this inverse relationship is to be found in Figure 10.

In Figure 11 the ratios are plotted on a scatter diagram. This diagram is a graphic representation of the observed relation between consumption ratios and price ratios during the period from 1890 to 1914. Our problem now is how best to adjust these observations in order to derive from them for any one year

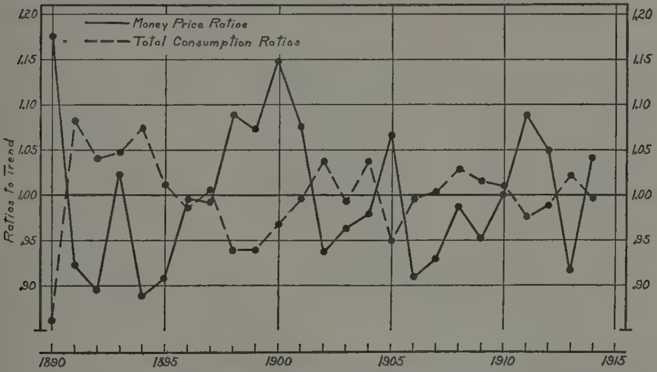

FIG. 10.—Inverse correlation between the trend ratios of the total consumption of sugar in the United States and the trend ratios of the wholesale money prices of sugar.

the *most probable* relation between the variations in the amount of sugar that is consumed and the corresponding variations in price. In other words, our problem is to pass a good continuous curve through these observations.

At this point two questions or difficulties present themselves: First, what function shall be fitted to the data? Second, how shall the difficulty connected with the choice of the independent variable be overcome? The second difficulty has already been discussed.[19] We have seen that it is effectively overcome when, in the fitting of the curve, both variables are consid-

[19] Cf. pp. 35–39.

ered as being subject to error. The first difficulty does not arise at all when the regression is linear, for the use of the straight line is then clearly indicated. When the regression is skew, it will be found that the advantages of using Professor Moore's "typical equation to the law of demand"[20] are such as not to

[20] H. L. Moore, "Elasticity of Demand and Flexibility of Prices," *Journal of the American Statistical Association*, March, 1922.

Professor Moore's equation is based on a generalized treatment of the coefficient of the elasticity of demand. Assuming that the *reciprocal* of this coefficient is a linear function of the amount demanded, he writes

$$\frac{dy}{dx}\cdot\frac{x}{y}=\alpha+\beta x \qquad \text{(i)}$$

where x stands for the amount demanded and y stands for the price per unit. The foregoing equation may be put in the form

$$\frac{dy}{y}=\alpha\frac{dx}{x}+\beta dx\,.$$

Integrating, he obtains

$$\log_e y=\alpha \log_e x+\beta x+\log_e A\,.$$

Or, passing from logarithms to absolute numbers,

$$y=Ax^{\alpha}e^{\beta x}\,. \qquad \text{(ii)}$$

Equation (i) was reached on the assumption that the reciprocal of the elasticity of demand is a linear function of x. Making the hypothesis a little more complex by assuming that

$$\frac{dy}{dx}\cdot\frac{y}{x}=\alpha+\beta x+\gamma x^2 \qquad \text{(iii)}$$

Professor Moore derives the equation

$$y=Ax^{\alpha}e^{\beta x+\frac{1}{2}\gamma x^2} \qquad \text{(iv)}$$

which is his "typical equation to the law of demand in a slightly more complex form."

The great advantage of the "typical equation" is that it enables us immediately to write down the equation descriptive of the coefficient of elasticity of demand (or of its reciprocal), for (i) appears in the exponent of (ii), and (iii) appears in the exponent of (iv).

The general forms (ii) or (iv), Professor Moore shows, also hold when the equations are desired in the *ratio form*. Thus (ii) may be written

$$Y=A'X^{\alpha'}e^{\beta' X}\,. \qquad \text{(v)}$$

When the mean values of X and Y are, practically, 1.0, the value of A' is $e^{-\beta}$ and consequently (v) becomes

$$Y=X^{\alpha'}e^{\beta'(X-1)}\,. \qquad \text{(vi)}$$

This is equation B, Figure 11.

make it worth our while to experiment much with other equations.

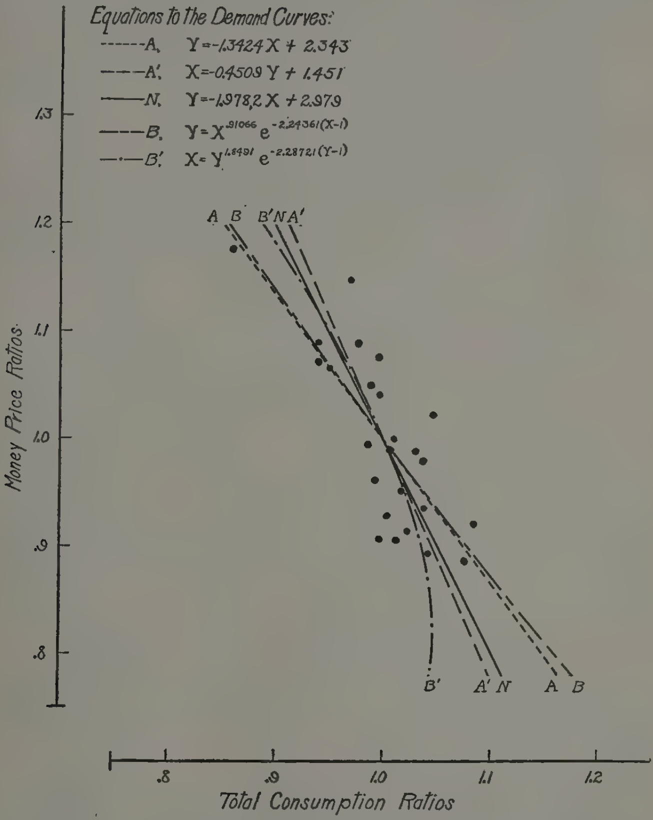

FIG. 11.—Scatter diagram of the trend ratios of the total consumption of sugar in the United States and the corresponding trend ratios of the wholesale money prices of sugar, and five fitted curves.

It is evident from the scatter diagram (Fig. 11) and from the usual tests for linearity of regression that the regression be-

fore us is quite linear. We take it, therefore, that the "best" demand curve within the limits of observation is given by the "normal line" N, or the "line of best fit." For comparative purposes, however, there are also graphed the straight lines A and A' and Moore's typical demand curves B and B'. The unaccented letters A and B are used to designate the curves giving the regression of Y on X, the accented letters A' and B' are used to designate curves giving the regression of X on Y.

It is also clear that if we were compelled to follow the usual practice of arbitrarily selecting either consumption or prices as the independent variable, we should have gotten better results by selecting the latter, for the curves A' and B' lie much closer to the "line of best fit" N than do the curves A and B.

Furthermore, it will be seen that in the problem before us nothing is gained through the use of the more complicated curve B or B' so long as one of the variables is considered to be free from errors of observation, and that when both variables are treated as being affected by errors, the straight line N gives as good a fit as can be expected.[21] (See also Fig. 12.) Of course, it might be argued that if the "typical equation" were also fitted on the assumption that both X and Y are subject to error, we should obtain a curve which is superior to the straight line N for purposes of extrapolation. Without stopping to consider the

[21] In comparing the straight lines with the other curves of Figure 11, allowance must be made for the fact that the former were fitted to the absolute values of the trend ratios, while the latter were fitted to the logarithms of these values, without making any allowance for the difference in weighting thus introduced. The same curve may sometimes give widely different results, depending upon whether it was fitted by minimizing the sum of the squares of the absolute figures or of their logarithms.

By properly weighting the logarithms of the trend ratios it is possible to fit the curves B and B' so as to make the sum of the squares of the absolute figures a minimum. At the time of fitting, however, we believed (perhaps erroneously) that the extra labor involved would not be worth while.

Perhaps it is also worth while to mention in this connection that the method of fitting which takes the errors in both variables into consideration can be applied not only to the straight line but to all curves which are reducible to linear form by logarithmic or other transformation.

mathematical difficulties which must be overcome if the newer method of curve-fitting is to be applied to the more complicated functions, we may reply that to extend *any* empirical demand

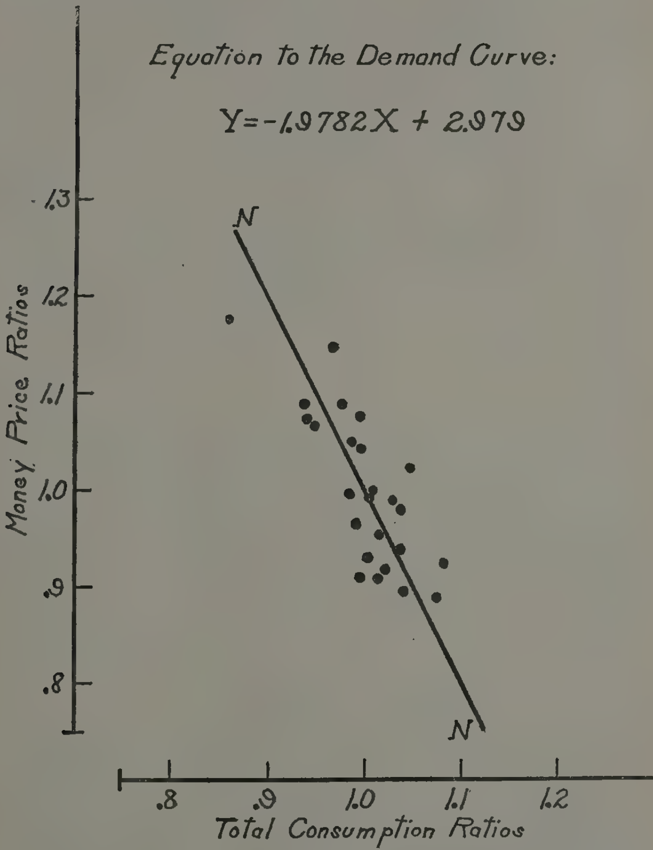

FIG. 12.—The "best" law of demand for sugar derived from the trend ratios of total consumption and the trend ratios of money prices.

curve much beyond the range of observation is not a safe procedure, as is evident from curve B' (Fig. 11), and that in most practical problems the extreme ends of the demand curve are of no importance anyway.[22] We therefore conclude that the "best" demand curve that may be derived from the data before us is the straight line N.

3. THE LAW OF DEMAND IN TERMS OF TREND RATIOS

The equation to our demand curve is

$$Y = -1.9782\,X + 2.9786 \tag{7}$$

the origin being at 0,0. This means that, based upon the experience from 1890 to 1914, an increase of one *point* (or, roughly, 1 per cent) in the (trend) ratio of consumption is correlated with a drop of 1.98 *points* in the (trend) ratio of the price, or that an increase of one *point* in the price ratio is associated with a decrease of only $1/1.9782 = 0.51$ of a *point* in the consumption ratio.

By means of this equation it is possible to estimate the probable change in the price corresponding to a given change in consumption, or the probable change in consumption corresponding to a given change in price. In 1914, the consumption of sugar in the United States amounted to 4,212,000 short tons. What was the probable price of sugar in the same year? The "normal" consumption or the "trend" of consumption in 1914 was 4,231,000 tons. (See Table II of Appendix II.) The consumption ratio was, therefore, $4{,}212{,}000 \div 4{,}231{,}000 = 0.996$. Substituting this value for X in equation (7) and solving for Y we find that $Y = 1.009$, or the probable price would be 1.009 of the "normal" or "trend" price. Since the trend price for 1914 was 4,493 cents (see Table II of Appendix II), this value of Y would give 4.533 as the probable price for 1914. The corresponding probable price obtained by the method of link relatives was, it will be recalled, 4.536 cents. The very close agreement between these two

[22] See Marshall, *Principles of Economics* (8th edition), p. 133 and note.

estimates is, of course, accidental. In like manner this equation can be used for estimating changes in consumption from given changes in price.

4. THE LAW OF DEMAND IN TERMS OF ABSOLUTE QUANTITIES

Equation (7) gives the most probable relation which existed between the consumption ratios and the price ratios of sugar during the interval 1890–1914. It is the law of demand for sugar in the ratio form. Professor Moore has shown,[23] however, that a simple transformation makes it possible to take the next step—the derivation of the law of demand for any one year in terms of absolute quantities. By definition, $X = x/T_x$ and $Y = y/T_y$. By making these substitutions in (7) the law of demand for sugar in the absolute form becomes

$$y = -1.9782\,(T_y/T_x)x + 2.9786\,T_y \tag{8}$$

which may also be written

$$x = -0.5055(T_x/T_y)y + 1.5057\,T_x\,. \tag{8a}$$

If, for example, it is desired to ascertain the law of demand for sugar in 1914 in a form convenient for estimating prices from consumption, all that we have to do is to find the trend of consumption for 1914, T_x, and the trend of prices for 1914, T_y, and substitute these values in (7). The values of T_x and T_y for any one year between 1890 and 1914 are given in Table II of Appendix II. Substituting the values of T_x and T_y for 1914 in (7) we obtain

$$y = -0.000{,}002{,}101\,x + 13.383\,.$$

That is to say, a change of 100,000 tons in consumption would, in 1914, have been associated with an inverse change in price of 0.21 of one cent. The corresponding figure derived by the method of link relatives (see equation [5]) is 0.19 of one cent. The present estimate is, however, more reliable, since the two series

[23] H. L. Moore, *op. cit.*, p. 16.

of trend ratios are more highly correlated than are the corresponding series of link relatives.

If, on the other hand, we should desire to write the same equation in a form more convenient for estimating consumption from price, all that we would have to do is to substitute the values T_x and T_y in (8*a*) instead of in (8), which, for 1914 becomes

$$x = -476{,}000\,y + 6{,}371{,}000\,.$$

The corresponding equations for 1904 and 1894 are

$$x = -310{,}000\,y + 4{,}500{,}000$$

and

$$x = -229{,}000\,y + 3{,}157{,}000$$

respectively. That is to say, a rise or fall in the price of sugar of one cent per pound would have decreased or increased consumption by 229,000 tons in 1894, by 310,000 tons in 1904, and by 476,000 tons in 1914. These estimates are to be preferred to those obtained by the method of link relatives and given on page 42, since the trend ratios on which they are based are more highly correlated than are the corresponding link relatives. For the same reason it is also fair to assume that the foregoing equations provide a better measure of the shifting of the demand curve from year to year than is provided by the corresponding equations derived by the method of link relatives (see p. 42).

5. COEFFICIENT OF THE ELASTICITY OF DEMAND

After the law of demand has been ascertained in one form or another—absolute quantities, link relatives, or trend ratios—it is a simple matter to find the coefficient of the elasticity of demand at any point on the demand curve. When the law of demand is given in terms of absolute quantities, then, by definition, $\eta = \frac{dx}{dy} \cdot \frac{y}{x}$, where x and y are absolute (observed) consumption and prices, respectively. We have proven on page 45 that when the law of demand is given in terms of link relatives, $\eta = \frac{dX}{dY} \cdot \frac{Y}{X}$,

where X and Y are the link relatives of consumption and prices, respectively. It can also be shown[24] that the same form holds when the law of demand is given in the trend-ratio form, i.e., when $X=\frac{x}{T_x}$ and $Y=\frac{y}{T_y}$. Applying this formula to the equations of the demand curves of Figure 11, and recalling that the equations to curves B and B' are particular forms of Professor Moore's "typical equation to the law of demand" which enable us immediately to write down the equation descriptive of the coefficient of elasticity of demand (see note 20, p. 56), we obtain

$$\left.\begin{aligned} \eta_A &= -(1/1.3424)Y/X \\ \eta_{A'} &= -0.4509\,Y/X \\ \eta_N &= -(1/1.9782)\,Y/X \\ \eta_B &= \;\;1/(0.91066-2.24361\,X) \\ \eta_{B'} &= \;\;1.8491-2.28721\,Y \end{aligned}\right\} \qquad (9)$$

where the subscripts A, A', etc., denote the demand curves from which these equations are derived. In Figure 13 these equations are graphed for a considerable range of values of X.

It will be seen that there is a considerable discrepancy between the results: (1) when different curves are used for the demand curve, and (2) when the assumption is made that either X or Y is free from error. This discrepancy, however, does not invalidate the method. It simply shows "the need of care in drawing conclusions that are based upon the numerical values of the coefficient of elasticity." In the words of Professor Moore,

> When different measures of degrees of elasticity are afforded by different types of curves, there is a perfectly satisfactory criterion which makes it possible to decide between different coefficients of elasticity: The coefficient is to be preferred which is deduced from the demand curve that fits the data with the highest degree of probability. The demand curve that fits best the data affords the best measure of the degree of elasticity of demand.[25]

[24] Moore, *op. cit.*, p. 19.

[25] Moore, *Economic Cycles*, p. 84.

We have seen that the demand curve that fits the data with the highest degree of probability is the line N (Figs. 11 and 12).

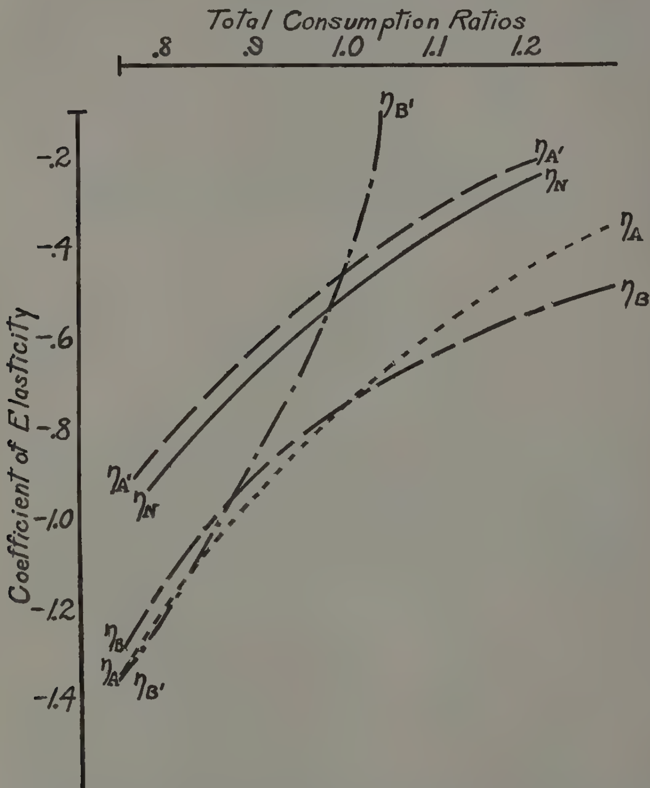

FIG. 13.—Elasticity of demand for sugar derived from the five demand curves of Figure 11.

We conclude, therefore, that the curve η_N (Fig. 13) affords the best measure of the degree of elasticity of demand, for it is deduced from the equation to the line N. From this curve it is

clear that the elasticity of demand is higher for low consumption (or high prices) than it is for high consumption (or low prices). For example, when $X=1$, that is to say, when the consumption of sugar in any year is "normal," or is equal to that indicated by the trend for the same year, then $\eta=-0.51$, or a reduction in the price of 1 per cent will increase consumption by 0.51 of one per cent. When $X=1.1$, that is to say, when the consumption for any year is already 10 per cent above "normal" or above that indicated by the trend for the same year, then $\eta=-0.37$, or a reduction of 1 per cent in the price will increase consumption by only 0.37 of 1 per cent. When $X=0.9$, that is to say, when consumption for any year is 10 per cent below "normal" or below that indicated by the trend for the same year, then $\eta=-0.67$, or a reduction of 1 per cent in price will increase consumption by 0.67 of one per cent.

6. COMPARISON WITH THE METHOD OF LINK RELATIVES

These figures naturally invite a comparison with the results obtained by the method of link relatives (see pp. 45–46). But before we can draw any conclusions from differences between coefficients of elasticity of demand, we must be sure that the coefficients relate to *comparable points* on the demand curve under consideration. That is to say, considering that the two methods have different interpretations, we must find the unit of link relative which corresponds to a given unit of trend ratio.

A comparison of Figure 12 or Figure 11 with Figure 5 suggests that the most comparable points on the two demand curves are the ones whose co-ordinates are given by the arithmetic means of the variables under consideration. We also know that deviations from these points will be comparable (assuming that they are measured in the same direction from the mean) if account is taken of the difference between the standard deviations of the series under comparison. The mean of the (total consumption) trend ratios is 1.0, and the mean of the (total consumption) link relatives is 1.0415. That is to say, a "normal"

consumption in terms of link relatives is one which exceeds the consumption for the previous year by 4.15 per cent.[26] The standard deviation of the (consumption) trend ratios is 0.0458 and the corresponding figure for link relatives is 0.0629. The two standard deviations are in the ratio of 1.0 to 1.3734, which means that a deviation of $\pm$ 1.0 unit from the mean of the trend ratios corresponds to a deviation of $\pm$ 1.3734 units from the mean of the link relatives. It follows, therefore, that (consumption) trend ratios of 0.9, 1.0, and 1.1 correspond to link relatives of 0.9042, 1.0415, and 1.1788, respectively. When the coefficient of elasticity of demand is computed for *comparable points* on the two demand curves, the results obtained by the method of link relatives do not differ materially from those obtained by the method of trend ratios. Thus, for the three trend ratios given above, the coefficients of the elasticity of demand are —0.67, —0.51, and —0.37, respectively; for the corresponding link relatives the coefficients are —0.72, —0.52, and —0.34 respectively.

Our conclusions regarding the elasticity of demand for sugar may, then, be summarized as follows: When the consumption for any year is "normal," that is to say, when it is equal to that indicated by the trend of consumption for the same year, or when it exceeds the consumption for the year immediately preceding by 4 per cent, then $\eta = -0.5$, or a decrease in price of 1 per cent will increase consumption by 0.5 of 1 per cent. The elasticity of demand under other conditions of consumption may be read off from the graph of η_N in Figure 13.

III. PARTIAL BIBLIOGRAPHY ON THE DETERMINATION OF LINES AND PLANES OF CLOSEST FIT

a) R. J. Adcock, "Note on the Method of Least Squares," *Analyst*, IV, No. 6 (November, 1877), 183; *ibid.*, V, No. 1 (January, 1878), 21; "A Problem in Least Squares," *ibid.*, V, No. 2

[26] In general, however, the arithmetic mean of the link relatives is not a good measure of the average annual increase.

(March, 1878), 53. The formula for *a* given by Adcock on p. 54 is incorrect "in consequence of an error which he committed in eliminating *b*." See p. 102, of reference (b) below.

b) Charles H. Kummell, "Reduction of Observation Equations which contain more than One Observed Quantity," *Analyst*, VI, No. 4 (July, 1879), 97.

c) Mansfield Merriman, "The Determination, by the Method of Least Squares, of the Relation between Two Variables, Connected by the Equation $Y=AX+B$, Both Variables Being Liable to Errors of Observation," *Report U.S. Coast and Geodetic Survey* (1890), p. 687. Cf. also *A Textbook on the Method of Least Squares* (1915), pp. 127, 128, 216, and 217, by the same author.

d) Karl Pearson, "On Lines and Planes of Closest Fit to Systems of Points in Space," *Philosophical Magazine*, 6th series, II (November, 1901), 559.

e) L. F. Reed, "Fitting Straight Lines," *Metron*, I (1921), No. 3 (I–IV), 54.

f) Corrado Gini, "Sull'interpolazione di una retta quando i valori della variabile indipendente sono effetti da errori accidentali," *ibid.*, p. 63. Cf. also Corrado Gini, "Considerazioni sull'interpolazione e la perequazione delle serie statistiche," *ibid.*, p. 3.

g) H. S. Uhler, "Method of Least Squares and Curve Fitting," *Journal of the Optical Society of America and Review of Scientific Instruments*, VII, No. 11 (November, 1923), 1043. "Determination of the Minimum Plane in Four-Dimensional Space with Respect to a System of Non-Coplanar Points," *Philosophical Magazine*, 6th series, XLIX (June, 1925), 1260. "Least Distance from a Point to a Linear (n-k) Space, Both in a Linear n-Space," *Annals of Mathematics*, 2d series, XXVII, No. 1 (September, 1925), 65.

h) Gaetano Pietra, "Interpolating Plane Curves," *Metron*, III (1924), No. 3–4 (I–II), 311.

i) H. Glauert, "The Determination of the Best Linear Rela-

tionship Connecting Any Number of Variables," *Philosophical Magazine*, 6th series, L (July, 1925), 205.

j) F. J. W. Whipple, "On the Best Linear Relation Connecting Three Variables," *ibid.*, 7th series, I (February, 1926), 378.

k) E. C. Rhodes, "On Lines and Planes of Closest Fit," *ibid.*, III (February, 1927), 357.

l) C. F. Roos and A. Oppenheim, "A Symmetric Method for Fitting Lines and Planes," presented to the American Mathematical Society, Nashville, Tennessee, December 29, 1927. For abstract see *Bull. Amer. Math. Soc.*, XXXIV, March–April, 1928, p. 140.

Adcock, Reed, Gini, and Pietra confine their treatment to the straight line, although Pietra also promises to determine the parameters of the ordinary parabola by this method in a forthcoming paper; all the other writers also treat of the fitting of the linear equation in three or more variables, and Kummell also attacks the problem of fitting any curve, though without success.

Adcock was the first to suggest that the criterion of "best fit" be that which minimizes the sum of the squares of the normals.

Kummell argues that it is even better to minimize the weighted sum, $\sum_{i=1}^{m} p_i(\delta x_i)^2 + \sum_{i=1}^{m} q_i(\delta y_i)^2$, where δx_i and δy_i are the respective corrections in x_i and y_i, and p_i and q_i are the *weights* of x_i and y_i which enter the same observation equation. He shows that the problem does not admit of a solution "unless we make a certain assumption with regard to the weights of the observed quantities." He therefore assumes "that the precisions of x and y which enter the same observation equation . . . . have a constant ratio." From this, however, it does not follow that they have equal weight, for, as he points out, "if the weight be defined as the reciprocal square of the probable error, it is clear that the unit in which a quantity is expressed makes a great difference in the weight to be assigned."

Kummell's paper, which appears to be unknown to all later writers except Merriman, who refers to it, contains practically all the results reached by them. His formulas, however, are not so convenient to apply as are some of those developed by the later writers. Uhler's paper on the straight line is to be particularly recommended for its full statement of the problem.

Roos and Oppenheim start out with the general assumption that "inasmuch as the problem of curve-fitting appears to be independent of the choice of co-ordinate system, ... the expression to be minimized should be invariant for translation, homogeneous strain and for rotation," and show that these conditions are sufficient to determine the parameters of the straight line and the plane. Their method, however, does not tell us the *signs* which must be attached to the direction cosines of the plane. This is a serious limitation in practical work.

Pearson approaches the fitting of the linear equation in three variables from the geometrical standpoint. He assumes that a *good* fit (even if not the *best* fit) would be obtained by minimizing the sum of the squares of the (perpendicular) distances of the observed points from the plane.

Uhler, in his 1925 paper (*Phil. Mag.*, June, 1925), extends the Pearsonian method to the fitting of a plane in four-dimensional space.

Glauert considers the slightly different problem in which the variables involved fall into two classes, one subject to errors of observation and the other known accurately, and develops equations similar to those of Pearson.

Whipple points out that there is an assumption in Glauert's (and also in Uhler's) method which is "really an instruction to choose the units of the several variables so that the probable errors will have the same numerical value"; he therefore develops another criterion of the "best" linear relation, considering only the case of three variables. It is he who introduced the term "line of mutual regression."

Rhodes briefly reviews Pearson's and Glauert's papers and

points out that "both these writers assume that the precision of the errors involved in the measurements of the different variables is the same." He therefore considers the problem without this assumption, which would apply to only a very restricted field. Rhodes shows, as Kummell did in 1879 (whose paper, however, is unknown to him), that the *general* solution of this problem involves two sets of unknown quantities: (1) the standards of measurement involved in measuring the several variables (u_i) and (2) the standard deviations of the errors involved in measuring the same variables (σ_i); and that the problem does not admit of a solution unless some assumption is made regarding these unknowns. If we consider all the u's to be equal, Rhodes' equations reduce to the equations obtained by Pearson for minimizing normal distances.

Regarding the line which minimizes the sum of the squares of the normals, Professor E. J. Moulton, of Northwestern University, writes in a letter to the author dated January 21, 1926:

"On comparing the problem with certain problems in mechanics, it develops that the solution is essentially at hand in the writings of L. Euler (1765), *"Theoria motus corporum solidorum seu rigidorum,"* and of J. P. M. Binet (1813), *Journal Ecole Polytechnique,* XVI, p. 47, and in most treatises on mechanics since those days."

The reader who is interested in methods other than the method of least squares is also referred to the following papers:

Francis Y. Edgeworth, "A New Method of Reducing Observations Relating to Several Quantities," *Philosophical Magazine,* 5th series, XXIV (August, 1887), 222; *ibid.,* XXV (March, 1888), 184. "The Use of Medians for Reducing Observations," *ibid.,* 6th series, XLVI (December, 1923), 1074. H. H. Turner, "On Mr. Edgeworth's Method of Reducing Observations Relating to Several Quantities," *ibid.,* 5th series, XXIV (December, 1887), 466. Norman Campbell, "The Adjustment of Observations," *ibid.,* 6th series (February, 1920), 177; (May, 1924), 816.

CHAPTER III

THE LAW OF DEMAND FOR SUGAR DERIVED FROM ADJUSTED DATA

I. THE METHOD OF LINK RELATIVES APPLIED TO THE ADJUSTED DATA

In the preceding chapter we have derived the law of demand for sugar from unadjusted data, i.e., from prices which have not been deflated by an index number of prices and from consumption which has not been reduced to a per capita basis. We have employed two separate methods—the method of link relatives and the method of trend ratios—and have shown that although both methods yield good results, the latter is superior to the former in the problem before us. The same methods may, however, also be applied to per capita consumption and real prices. Would the use of these adjusted series yield still better results? The first part of this chapter will contain an answer to this question in so far as it applies to the method of link relatives. The second part will deal with the method of trend ratios. As most of the problems which arise in the application of these methods have already been discussed, they will be touched upon only slightly in the following pages.

In Table I A of Appendix II, there are recorded, for the period of 1890–1914, the annual per capita consumption of sugar, the corresponding real prices, the link relatives of per capita consumption, and the link relatives of the real prices. The coefficient of correlation between the two series of link relatives is $r = -0.67 \pm 0.08$. The corresponding coefficient of correlation derived from the unadjusted data is[1] $r = -0.68 \pm 0.07$. In the problem before us, then, there is no advantage to be derived from the use of adjusted data. It is desirable, however-

[1] Cf. p. 35.

ever, to complete the computations and derive the law of demand for sugar from per capita consumption and real prices as we have derived it from total consumption and money prices.

A graphic representation of the negative correlation existing between the two series of link relatives is to be found in Figure 14.[2] The scatter diagram of the link relatives is shown in Figure 15. (Compare Fig. 14 with Fig. 4 and Fig. 15 with Fig. 5.)

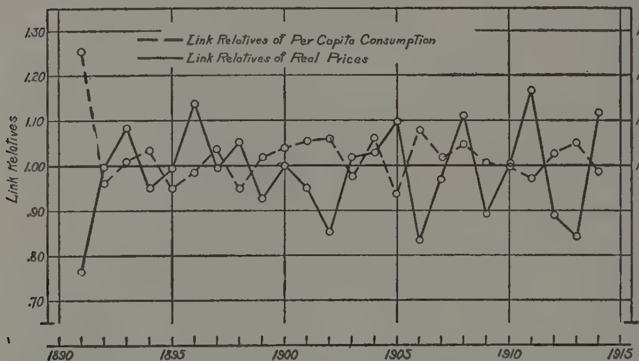

FIG. 14.—Inverse correlation between the link relatives of the per capita consumption of sugar in the United States and the link relatives of the real wholesale prices of sugar.

I. THE LAW OF DEMAND IN TERMS OF LINK RELATIVES

In Figure 15 there are also shown the two lines of regression, A and A', and the "line of best fit," N. The equation to the latter is

$$Y = -2.0817\,X + 3.113 \tag{10}$$

the origin being at 0,0. This equation tells us that, based upon the experience from 1890 to 1914 inclusive, an increase or de-

[2] All data relating to per capita consumption and real prices are represented by rings ○ to distinguish them from the data relating to total consumption and money prices, which are represented by closed circles ●.

crease of one *point* (or, roughly, 1 per cent) in the link relative of per capita consumption is, on the average, associated with a

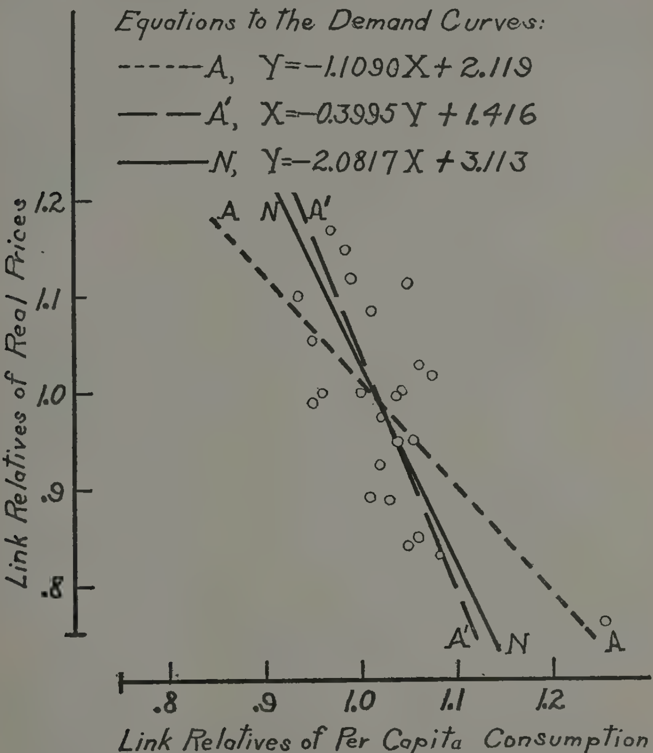

FIG. 15.—Scatter diagram of the link relatives of the per capita consumption of sugar and the corresponding link relatives of the real wholesale prices of sugar, with the lines of regression (A and A') and the "line of best fit" (N).

fall or rise of 2.08 *points* in the link relative of the real price. Another way of saying the same thing is that a change of one *point* in the link relative of the real price is, on the average, asso-

ciated with an inverse change of $1/2.0817 = 0.48$ of one *point* in the link relative of per capita consumption.

This equation enables us to estimate the probable change in the real price corresponding to a given change in the per capita consumption, or the probable change in the per capita consumption corresponding to a given change in the real price. In 1913 the per capita consumption of sugar in the United States was 85.4 pounds. In 1914 the consumption was 84.3 pounds. What was the probable real price of sugar in 1914? The change in the per capita consumption as measured by the link relative was $84.3/85.4 = 0.987$ (see Table I A of Appendix II). Substituting this value for X in equation (10) and solving for Y, we find that the probable real price would be 1.058 of the real price for the preceding year. Since the real price for 1913 was 3.730 cents, this value of Y would give 3.947 cents as the probable real price for 1914. The observed real price was 4.166 cents—a difference of 5 per cent. In like manner we can estimate the probable change in consumption corresponding to a given change in price.

2. THE LAW OF DEMAND IN TERMS OF ABSOLUTE QUANTITIES

Equation (10) is the law of demand for sugar in terms of the *link relatives* of the per capita consumption and the *link relatives* of the real prices. As in the case of equation (4), however, a simple transformation makes it possible to derive the law of demand for any one year in terms of *absolute* per capita consumption and real prices. By definition, $X = \frac{x}{x_0}$, and $Y = \frac{y}{y_0}$, where x_0 and y_0 are respectively the per capita consumption and the real prices in the preceding or base year. By making these substitutions in (10), the law of demand in the absolute form becomes

$$y = -2.0817\ (y_0/x_0)x + 3.113y_0 \qquad (11)$$

which may also be written

$$x = -0.4804\ (x_0/y_0)y + 1.495x_0\,. \qquad (11a)$$

These equations are equivalent to (5) and (5*a*) respectively, and are to be used in the same way.

As an example, suppose that it is desired to have the law of demand for sugar in 1914 in a form convenient for estimating price from per capita consumption. From Table I A of Appendix II, we find that the per capita consumption (x_0) and the real price (y_0) for the previous year, or 1913, were 85.4 and 3.730, respectively. Substituting these values of x_0 and y_0 in (11), we obtain

$$y = -0.09092\,x + 11.611$$

as the law of demand for sugar in 1914. This equation gives the probable real price corresponding to *any* given per capita consumption in that year. The observed per capita consumption in 1914 was 84.3 pounds. What was the probable real price? Substituting this value for x in the foregoing equation, we obtain 3.947 cents as the probable real price in 1914. The observed real price was 4.166 cents—a difference of 5 per cent.

Suppose now that it is desired to have the law of demand for 1914 in a form more convenient for estimating per capita consumption from real prices. Substituting the values given above for x_0 and y_0 in (11*a*), we obtain

$$x = -10.999\,y + 127.67$$

as the desired equation. By way of comparison we give the corresponding equations for 1904 and 1894. These are

$$x = -7.242\,y + 106.00$$

and

$$x = -5.641\,y + 96.28$$

respectively. That is to say, a rise or fall in the real price of sugar of one cent per pound would have decreased or increased the *per capita consumption* by 5.6 pounds in 1894, by 7.2 pounds in 1904, and by 11.0 pounds in 1914. Corresponding equations may, of course, be derived for all the other years between 1890

and 1914. Such equations provide a quantitative measure of a phenomenon in which economists are greatly interested—the degree to which the "demand schedule" changes from year to year. In the second part of this chapter, however, a more satisfactory method of determining these equations will be presented.

3. COEFFICIENT OF THE ELASTICITY OF DEMAND

From equation (10), the coefficient of the elasticity of demand is

$$\eta=\frac{dX}{dY}\cdot\frac{Y}{X}=-\frac{1}{2.0817}\frac{Y}{X}=-0.4804\frac{Y}{X}\,. \tag{12}$$

When $X=1$, that is to say, when there is no change in the per capita consumption between any two consecutive years, then $\eta=-0.4804\ (1.0313)=-0.50$, or a reduction in price of 1 per cent will increase consumption by 0.50 of 1 per cent. But the "normal" condition of per capita consumption during the period 1890–1914 was characterized by an average annual increase of about 2 per cent. (The means of the link relatives are $\overline{X}=1.0215$, $\overline{Y}=0.9864$.) When the per capita consumption for any year is "normal," that is to say, when it exceeds the consumption for the previous year by 2 per cent, then $\eta=-0.46$, or a decrease in price of 1 per cent will increase consumption by 0.46 of 1 per cent. Had we arbitrarily selected line A or line A' for our law of demand, we should have obtained as the corresponding value for η either -0.87 or -0.39, respectively.

II. THE METHOD OF TREND RATIOS APPLIED TO THE ADJUSTED DATA

In Figure 16 there is graphed, for the period of 1890–1914, the observed trend of the per capita consumption of sugar. In for our law of demand, we should have obtained as the corresponding real prices. With each observed trend there are also shown four fitted trends—a straight line, a quadratic, a cubic,

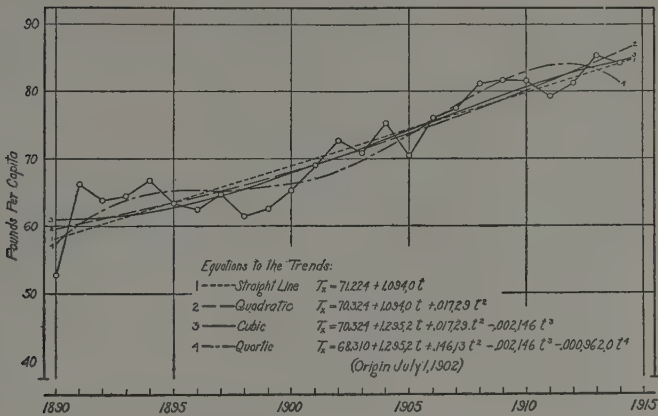

FIG. 16.—Per capita consumption of sugar in the United States, 1890–1914, and four fitted trends.

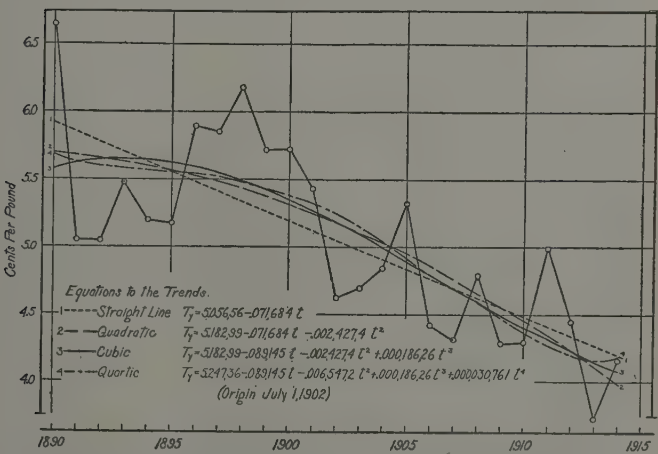

FIG. 17.—Real prices of refined sugar at New York, 1890–1914, and four fitted trends.

and a quartic.[3] It is interesting to observe that while the per capita consumption of sugar increased from 52.8 pounds in 1890 to 84.3 pounds in 1914, the real price declined from 6.643 cents to 4.166 cents during the same period.

I. POSSIBLE OBJECTIONS

In view of the inverse secular movements of per capita consumption and real prices, it may be asked, "Why have recourse to trend ratios at all? Is not the decline in the real price of sugar the *cause* of the increase of the per capita consumption? Why not correlate the figures as they stand?" Indeed, the degree of correlation between the observed per capita consumption and observed real prices is higher than that found between any of the series of trend ratios or link relatives hitherto observed, r being -0.90 ± 0.03. The position taken in this study, however, is that the existence of a trend in any series—per capita consumption or total consumption, real prices or money prices—ought to make us suspect the existence of disturbing factors which might produce a spurious correlation. (One obvious factor which may cause a spurious correlation is the growing taste for sugar.) The upward trend of the per capita consumption of sugar during the period under consideration *may* have been caused by the downward trend of the real prices. It is safer to assume, however, that there is no bona fide, organic relation between the trends. For if it is reasonable to expect a bona fide relation between the two trends, it is also reasonable to expect that the elasticity of demand derived from these series would be of the same order of magnitude as those obtained by the three methods described above—say, -0.5. But a computation shows that the observed per capita consumption and the observed real prices (both variables uncorrected for trend) are connected by the equation.

$$y = -0.07336\,x + 10.2820 \qquad (13)$$

[3] In the equations to these trends, x = observed per capita consumption in pounds; y = observed real price in cents (B.L.S. index for 1900–1909 = 1.00); T_x = per capita consumption computed from trend; T_y = real price computed from trend; t = time in years, the origin being at July 1, 1902.

where x = per capita consumption in pounds, and y = real prices in cents, the origin being at 0,0. According to this equation, the coefficient of the elasticity of demand at the mean of the variables ($\bar{x}=71.224$, $\bar{y}=5.057$) is $\eta=-0.97$. That is to say, the elasticity of demand derived from the series under consideration is 100 *per cent greater* than any of those derived thus far. We conclude, therefore, that the trends must be eliminated, as their retention introduces an element of spurious correlation.

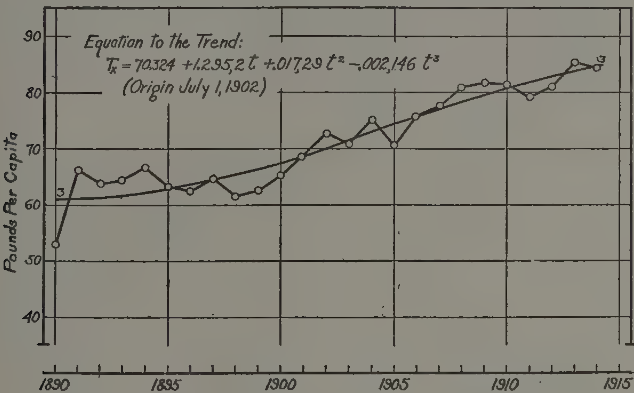

FIG. 18.—The "best" trend of the per capita consumption of sugar in the United States, 1890–1914.

In other words, the use of per capita consumption and real prices does not, in the problem at hand, do away with the necessity of having recourse to the method of trend ratios or to a similar method.

2. THE SELECTION OF A SUITABLE TREND

In selecting a suitable trend for each of our adjusted series (see Figs. 16 and 17), we are in a position to benefit from our experiments with the trends of the unadjusted series.[4] Accordingly, we select, by inspection, the third-degree parabolas as giving the "best" compromises between smoothness and goodness of fit. Figure 18 shows the "best" trend of per capita consump-

[4] See pp. 47–54 and Figs. 6–9.

tion, and Figure 19 shows the "best" trend of real prices, for the period under consideration.

In Table II A of Appendix II, there are recorded the observed per capita consumption and the observed real prices of sugar; the "normal" per capita consumption and the "normal" real price as computed from the most satisfactory trends; and

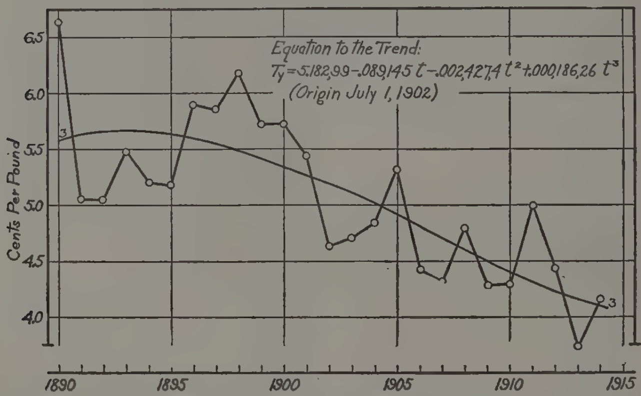

FIG. 19.—The "best" trend of the real wholesale prices of refined sugar at New York, 1890–1914.

the consumption ratios (X) and the price ratios (Y). The coefficient of correlation between the two series of trend ratios is $r = -0.80 \pm 0.05$. A graphic presentation of the inverse movement of the trend ratios is given by Figure 20. As the correlation between the corresponding trend ratios of *total* consumption and *money* prices is, as we have seen, just as high ($r = -0.78 \pm 0.05$), we conclude that, in the problem before us, no significantly more accurate law of demand for sugar may be derived from per capita consumption and real prices than that which we have derived from total consumption and money prices. (Compare Fig. 20 with Fig. 10.) In view, however, of our previous

remarks[5] concerning the importance of eliminating the trend even when we use adjusted data, it is desirable to proceed with the computations and derive the demand curve and the elasticity of demand from the trend ratios of the adjusted data. The results of the computations are exhibited in Figures 21, 22, 23.

3. COMPARISON OF SEVERAL TYPES OF DEMAND CURVES

In Figure 21 there are shown the results obtained by fitting a straight line and two forms of Moore's "typical equation to the

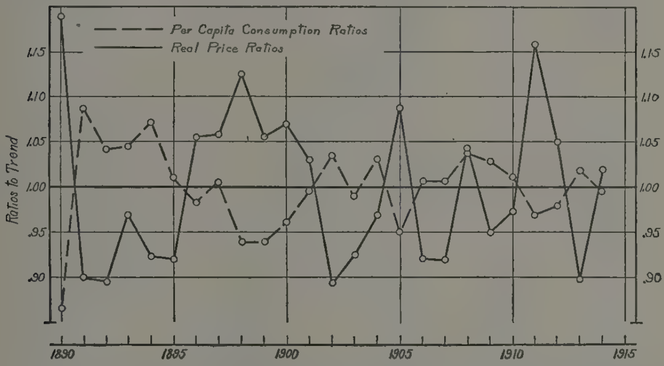

FIG. 20.—Inverse correlation between the trend ratios of the per capita consumption of sugar in the United States and the trend ratios of the real wholesale prices.

law of demand" to our data: (1) when it is assumed that Y alone is subject to error (curves A, B, C), and (2) when it is assumed that X alone is subject to error (curves A', B', C'). For comparative purposes there is also shown the straight line (line N) obtained by assuming that *both* variables are subject to error.

A study of the curves of Figure 21 suggests the following conclusions:

[5] Cf. pp. 78–79.

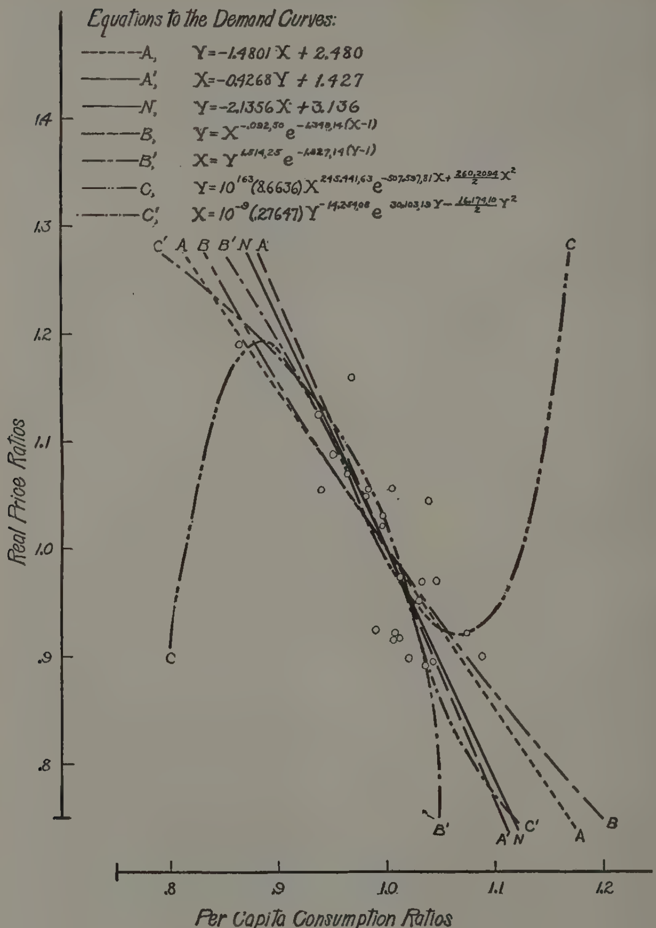

FIG. 21.—Scatter diagram of the trend ratios of the per capita consumption of sugar in the United States and the corresponding trend ratios of the real wholesale prices of sugar, and seven fitted curves.

1) When the regression is practically linear, as in the problem at hand, the use of more or less complex equations to repre-

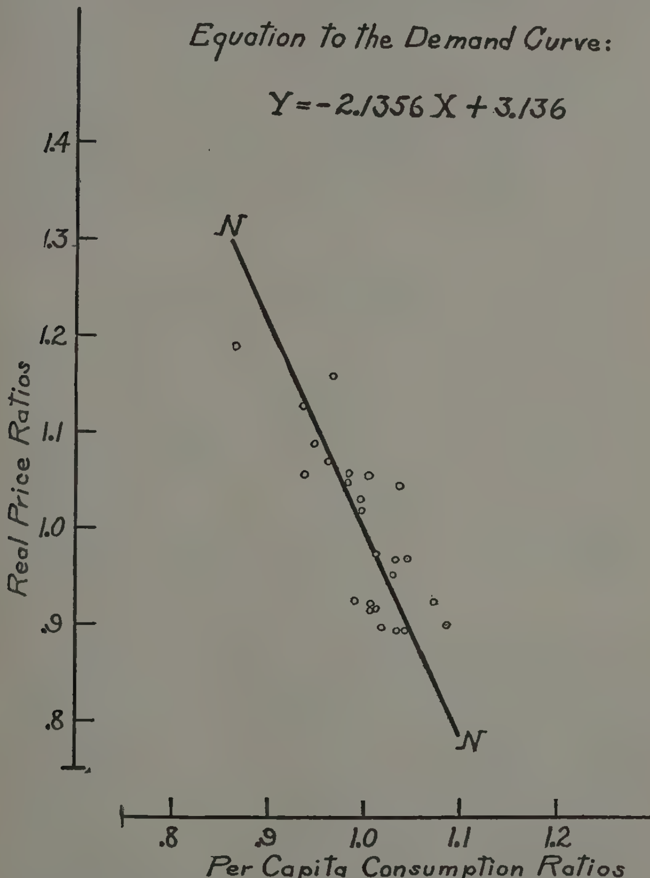

FIG. 22.—The "best" law of demand for sugar derived from the trend ratios of per capita consumption and the trend ratios of real prices.

sent the law of demand may give impossible results.[6] (See curve C, Fig. 21, and curves η_c and $\eta_{c'}$, Fig. 23.)

2) The effect of assuming that only one of the variables is subject to error may be not only to modify the *slope* of the demand curve, as in the case of A and A', but also to change the direction of its *concavity*, as in the case of B and B'. (Curve B is concave upward, while curve B' is concave downward.)

3) The "best" demand curve is given by the "line of best fit" (line N) which was fitted on the assumption that both series of trend ratios are affected by errors. It is shown separately in Figure 22.

4. THE LAW OF DEMAND IN TERMS OF TREND RATIOS

The equation to our demand curve (line N) is:

$$Y = -2.1356\,X + 3.136 \tag{14}$$

the origin being at 0,0. This equation tells us that, based on the experience of 1890–1914, an increase of one *point* (or, roughly, 1 per cent) in the trend ratio of per capita consumption is, on the average, associated with a decrease of 2.14 *points* in the trend ratio of the real price, or that an increase of one *point* in the price ratio is, on the average, associated with a decrease of $1/2.1356 = 0.47$ of one *point* in the consumption ratio. The cor-

[6] In comparing the straight lines with the other curves of Figure 21, allowance must, however, be made for the fact that the former were fitted to the absolute values of the trend ratios, while the latter were fitted to the logarithms of these values, without making any allowance for the difference in weighting thus introduced. The same curve may sometimes give widely different results depending upon whether it was fitted by minimizing the sum of the squares of the absolute figures or of their logarithms.

By properly weighting the logarithms of the trend ratios it is possible to fit the curves B and B' and C and C' so as to make the sum of the squares of the absolute figures a minimum. At the time of fitting, however, we believed (perhaps erroneously) that the extra labor involved would not be worth while.

Perhaps it is also worth while to mention in this connection that the method of fitting which takes the errors in both variables into consideration can be applied not only to the straight line but to all curves which are reducible to linear form by logarithmic or other transformation.

responding figures derived from the trend ratios of *total* consumption and *money* prices are 1.98 and 0.51 *points* respectively.

This equation enables us to obtain a better estimate of the probable real price from a given per capita consumption than that provided by equation (10), for the reason that the trend ratios are more highly correlated than the corresponding link relatives. This equation corresponds to equation (7), which gives the relation between the trend ratios of money prices and total consumption, and is to be used in the same way. For example, suppose that it is desired to find the probable real price of sugar in 1914. Referring to Table II A of Appendix II, we see that the ratio of the observed per capita consumption for 1914 to the trend of per capita consumption for the same year was 0.996. Substituting this value for X in (14) we find that $Y=$ 1.009, or the probable price would be 1.009 of the "normal" or "trend" price. Since the trend price for 1914 was 4.086 cents, this value of Y would give 4.123 cents as the probable real price for 1914. The observed real price was 4.166 cents, which differs from the probable price by 1 per cent. The corresponding probable price obtained by the method of link relatives, it will be recalled, was 3.947 cents. In like manner we can estimate the probable change in per capita consumption corresponding to a given change in the real price.

5. THE LAW OF DEMAND IN TERMS OF ABSOLUTE QUANTITIES

Equation (14) gives the law of demand in the *ratio form*. That is to say, $X=x/T_x$ and $Y=y/T_y$ where x=observed per capita consumption; y=observed real prices; T_x=trend of per capita consumption; and T_y=trend of real prices. It was shown elsewhere,[7] however, that a simple transformation makes it possible to derive the law of demand for any one year in terms of *absolute quantities*. All that we have to do is to substitute for X and Y in (14) their values as given above. We obtain

$$y=-2.1356\ (T_y/T_x)x+3.136T_y \qquad (15)$$

[7] Cf. p. 61.

or, with y as the independent variable,

$$x = -0.4683\,(T_x/T_y)y + 1.468T_x\,. \tag{15a}$$

(Compare [15] and [15*a*] with [8] and [8*a*], respectively.) In order, therefore, to ascertain the law of demand for any given year, it is necessary to find the values of T_x and T_y for that year and to substitute these values in (15) or (15*a*). (These values are given in Table II A of Appendix II.) The former equation is more convenient for estimating prices from consumption; the latter is more convenient for estimating consumption from price. These equations are equivalent to (8) and (8*a*) and are to be used in the same way.

Suppose, for example, that it is desired to have the law of demand for 1914 in a form convenient for estimating price from per capita consumption. From Table II A of Appendix II we find that T_x and T_y for 1914 amounted to 84.6 and 4.086, respectively. Substituting these values of T_x and T_y in (15), we obtain

$$y = -0.1031\,x + 12.814$$

as the law of demand for sugar in 1914. This equation gives the probable real price corresponding to *any* given per capita consumption in that year. The actual per capita consumption in 1914 amounted to 84.3 pounds. Substituting this value for x in the foregoing equation and solving for y, we obtain $y=4.123$, or the probable real price in 1914 was 4.123 cents. The observed real price was 4.166—a difference of 1 per cent. Such a close agreement between the probable price and the observed price is, of course, accidental. Similar equations may be obtained for all other years between 1890 and 1914.

Suppose now that it is desired to have the law of demand for 1914 in a form more convenient for estimating consumption from price. Substituting the values given above for T_x and T_y in (15*a*), we obtain

$$x = -9.696\,y + 124.22$$

as the desired equation. The corresponding equations for 1904 and 1894 are

$$x = -6.843\,y + 107.19$$

and

$$x = -5.160\,y + 91.33$$

respectively. In other words, a rise or fall in the real price of sugar of one cent per pound would have decreased or increased consumption by 5.2 pounds in 1894, by 6.8 pounds in 1904, and by 9.7 pounds in 1914. The corresponding figures obtained by the method of link relatives (see p. 75) are 5.6, 7.2, and 11.0 pounds, respectively. These estimates are, however, less reliable than those which we have just obtained through the use of equation (15*a*), as the link relatives are less highly correlated than the corresponding trend ratios.

Equations like the foregoing may, of course, be derived for the other years between 1890 and 1914. They provide a quantitative measure of the degree to which the "demand schedule" has shifted from year to year as a result of dynamic changes.

6. COEFFICIENT OF THE ELASTICITY OF DEMAND

We have shown elsewhere[8] that after the law of demand has been ascertained in one form or another—absolute quantities, link relatives, or trend ratios—it is a simple matter to find the coefficient of the elasticity of demand at any point on the demand curve. It can easily be shown that when the law of demand is given in the form of trend ratios, $\eta = \frac{dX}{dY} \cdot \frac{Y}{X}$, where X and Y are the trend ratios of consumption and prices, respectively. Applying this formula to the equations of the demand curves of Figure 21, and recalling that curves B, B', C, and C' are different forms of Professor Moore's "typical equation to the law of demand," which enables us immediately to

[8] Cf. pp. 45–46, 62–65, and 76.

write down the equations descriptive of the coefficient of elasticity of demand (see note 20, p. 56), we obtain

$$\left.\begin{aligned}
\eta_A &= -(1/1.4801)Y/X \\
\eta_{A'} &= -0.4268\ Y/X \\
\eta_N &= -(1/2.1356)Y/X \\
\eta_B &= -1/(0.0925+1.34914\ X) \\
\eta_{B'} &= 1.51425-1.92714\ Y \\
\eta_C &= 1/(245.44163-507.59781X+260.20940X^2) \\
\eta_{C'} &= -14.25408+30.10319Y-16.17410Y^2
\end{aligned}\right\} \qquad (16)$$

where the subscripts A, A', etc., denote the demand curves from which these equations are derived. In Figure 23 these equations are graphed for a considerable range of values of X.

A glance at the curves in Figure 23 is sufficient to warn us that great care must be exercised in drawing conclusions that are based upon the numerical values of the coefficient of elasticity of demand. The values of these coefficients fluctuate widely according to the type of curve that is selected to represent the law of demand and the method of fitting that is employed. Demand curves which look very much alike (curves A and B, or curves A' and N, Fig. 21) may yield radically different values for the coefficient of elasticity of demand (curves η_A and η_B, or $\eta_{A'}$ and η_N, Fig. 23). It is highly important, therefore, to find the demand curve that fits the data with the highest degree of probability. In this connection the method of fitting curves which considers both variables as being subject to errors of observation is of the greatest importance.

We have seen that the demand curve which fits the data with the highest degree of probability is the line N (Figs. 21 and 22). We conclude, therefore, that the curve η_N (Fig. 21) affords the best measure of the coefficient of elasticity of demand, for it was deduced from the equation to the line N.

From this curve it is clear that the elasticity of demand for sugar is higher (in absolute value) for low consumption (or

high prices) than it is for high consumption (or low prices). For example, when $X=1$, that is to say, when the per capita

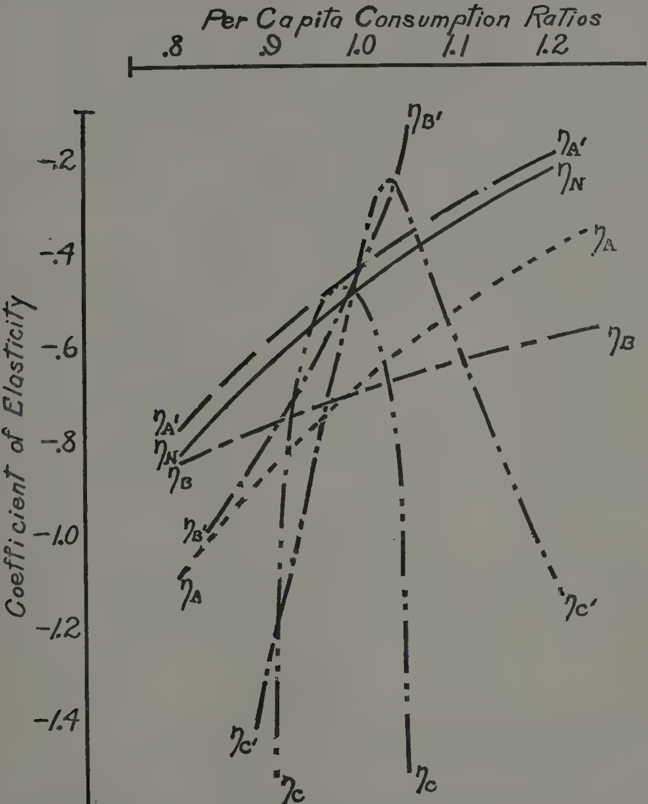

FIG. 23.—Elasticity of demand for sugar derived from the seven curves of Figure 21.

consumption of sugar for any year is "normal," or is equal to that indicated by the trend for the same year, then $\eta=-0.47$,

or a reduction in the (real) price of one per cent will increase consumption by 0.47 of 1 per cent. When $X=1.1$, that is to say, when consumption for any year is 10 per cent above "normal" or above that indicated by the trend for the same year, then $\eta=-0.33$, or a reduction of 1 per cent in price will increase consumption by only 0.33 of 1 per cent. When $X=0.9$, that is to say, when the consumption for any year is 10 per cent below "normal" or below that indicated by the trend for the same year, then $\eta=-0.63$, or a reduction of 1 per cent in price will increase consumption by 0.63 of 1 per cent.

When the law of demand for any one year is expressed in terms of absolute quantities, the coefficient of elasticity of demand is, of course, given by the formula $\eta=\frac{dx}{dy}\cdot\frac{y}{x}$, where x is the absolute consumption and y is the absolute price. By way of illustration, we may determine what the elasticity of demand would have been in 1914 had the per capita consumption in that year amounted to only 80 pounds instead of the observed 84.3 pounds. The law of demand for sugar in 1914 (see p. 86) is given by the equation

$$y=-0.1031\,x+12.814\,.$$

The probable price corresponding to a consumption of 80 pounds is, by this equation, equal to 4.566 cents. Hence

$$\eta=\frac{dx}{dy}\cdot\frac{y}{x}=-\frac{1}{0.1031}\,\frac{4.566}{80}=-0.553,$$

or the coefficient of elasticity would have been -0.55.

7. COMPARISON WITH THE METHOD OF LINK RELATIVES

It was pointed out elsewhere (see pp. 65–66) that before we can compare coefficients of elasticity of demand derived by different methods, we must be sure that they relate to the *comparable points* on the demand curves under consideration. It was also shown that a point on one demand curve will be comparable with a point on another demand curve if the deviations of the

two points from their respective means have the same sign and if they are in the same ratio as the standard deviations of the two series under consideration. The mean of the (per capita consumption) trend ratios is 1.0, and the mean of the (per capita consumption) link relatives is 1.0215. That is to say, a "normal" consumption in terms of link relatives is one which exceeds the consumption for the previous year by 2.15 per cent. The standard deviation of the (consumption) trend ratios is 0.0461 and the corresponding figure for (consumption) link relatives is 0.0626. The two standard deviations are in the ratio of 1.0 to 1.3579, which means that a deviation of $\pm$ 1.0 unit from the mean of the trend ratios corresponds to a deviation of $\pm$ 1.3579 units from the mean of the link relatives. It follows, therefore, that (consumption) trend ratios of 0.9, 1.0, and 1.1 correspond to link relatives of 0.8855, 1.0215, and 1.1575, respectively. When the coefficient of elasticity of demand is computed for *comparable points* on the two demand curves, the results obtained by the method of link relatives (Equation [12]) are not materially different from those obtained by the present method (η_N Equation [16]). Thus, the values of the coefficient of the elasticity of demand for the three trend ratios given above are —0.63, —0.47, and —0.33, respectively. The values of the coefficient for the corresponding link relatives are —0.69, —0.46, and —0.29, respectively. It will be recalled that the two methods also yielded practically identical results when they were applied to unadjusted data, i.e., to total consumption and money prices.

This brings us to the end of the investigation of the law of demand for sugar as derived from adjusted data. We have seen that, in the problem before us, the results obtained from the adjusted data are practically identical with those derived from the unadjusted data. It does not follow, however, that in another problem no advantage will be gained through the use of per capita consumption and real prices. All statistical devices are to be valued according to their efficacy in the immediate

problem under consideration. If, for example, it is found that "deflated" prices give better results than observed money prices, the use of such adjusted prices needs no other justification for the purpose in view.

III. SUMMARY AND CONCLUSIONS RELATING TO DEMAND

In chapter i we have briefly examined the concept of elasticity of demand, the assumptions that are inherent in the theoretical classical law of demand, and the difficulties that are met with in the derivation of concrete, statistical laws of demand. We have shown how these difficulties may be overcome, how both static and dynamic laws of demand may be derived, by modern statistical methods.

Proceeding to chapter ii, we have applied some of the simpler methods of the problem of finding the law of demand for sugar. We have derived this law of demand in four different ways and have found that the different methods give approximately the same value (—0.5) for the coefficient of elasticity of demand under normal conditions of consumption. That is to say, we have established the fact that, under normal conditions, an increase of 1 per cent in the price of sugar is associated with a decrease in consumption of only 0.5 of 1 per cent. We have defined "normal conditions" and have shown that even when the conditions of consumption differ considerably from normal, the coefficient of elasticity of demand is still numerically less than 1.0. The demand for sugar may, therefore, be designated as quite inelastic. In the course of our computations we have had occasion to call attention to the two difficulties that are inherent in the method of trend ratios. There is, first, the difficulty of selecting satisfactory trends from which to compute "normal" consumption and "normal" prices; there is, second, the difficulty of taking into account the "errors" or deviations in *both* variables when we proceed to fit the demand curve. We trust that the statistical experiments which have been suggested by

these difficulties, incomplete as they are, will be of some help to other workers in the same field.

In concluding Part I of this study we may call attention to some phases of the statistical law of demand which have not perhaps been sufficiently emphasized in the preceding pages.

1. The law of demand for sugar derived in this study is, strictly speaking, applicable to the period 1890–1914 only. Whether the same law holds today must be determined from a comparison of the present forces behind the demand and the supply of sugar with those which were in operation during the twenty-five years before 1915. In any event, the trend of consumption and the trend of prices for any year after 1914 may *not* be obtained, by extrapolation, from the corresponding trends for the period 1890–1914. The trends which are used in this study are satisfactory only within the limits of observation. They may give impossible results if extrapolated beyond these limits. The problem of deriving trends which may be used for extrapolation as well as interpolation has not been considered in this study. Of course, the whole difficulty of extrapolation might be avoided by using the law of demand based on link relatives or percentage changes. We have seen, however, that, with the data before us, this demand curve is not so satisfactory as that based on the trend ratios.

2. The coefficient of the elasticity of demand for sugar has been derived *from* the demand curve. As we have stated elsewhere,[9] it is our conviction that the accepted definition of the coefficient of elasticity of demand *presupposes* a knowledge of the demand curve, and that any attempt to derive the elasticity of demand for a commodity without first deriving the demand curve is apt to lead to difficulties. Professor Lehfeldt's derivation of the elasticity of demand for wheat[10] illustrates this difficulty. See Appendix I.

[9] Cf. p. 7.

[10] R. A. Lehfeldt, "The Elasticity of Demand for Wheat," *Economic Journal*, June, 1914.

3. The demand curves which were finally selected in this study are those which were fitted on the assumption that the reason why any point fails to fall on the curve is that it is subject to a horizontal as well as to a vertical "error." The "errors" with which we have to deal are not only, or even mainly, the *accidental* errors which are due to no known cause of systematic or constant error and which play such an important rôle in the theory of least squares. *But they are treated as though they were true accidental errors.* That is to say, we first eliminate such constant or systematic "errors" as may be eliminated through the use of index numbers, trend ratios, or link relatives. (Such constant or systematic "errors" are due, as a rule, to population growth and to changes in the general price level.) We are quite certain that there are others still, but we can not measure them. We therefore *assume* that they are eliminated by the graduation process involved in fitting the demand curve.[11]

4. The law of demand derived in this study is a *dynamic* law; it describes in summary form the "routine of change" of an important economic phenomenon. It is the dynamic law of demand in a simple form (link relatives or trend ratios). It is quite different from the static law of demand of the classical writers. "The statical law may be only approached (see equations 3 and 3*a*) but never realized in inductive investigation."[12]

5. The ease with which the general law of demand in terms of trend ratios or link relatives may be converted into a law of demand for any one year in terms of absolute quantities is of theoretical as well as practical importance. In the determination of the effects of a tax on the conditions of supply and in many related problems, it is necessary to have the cost or supply curve as well as the demand curve of the commodity in question. But the ordinary cost curve, for reasons which need not be discussed here, relates to a given year or date, and varies

[11] See Henri Poincaré, *Science and Hypothesis,* chap. xi, "The Calculus of Probabilities," section on "The Theory of Errors."

[12] Professor H. L. Moore in a letter to the present writer.

greatly from year to year. To compare such a cost curve, which is expressed in terms of absolute quantities, with a demand curve of the ratio or link-relative form is, of course, impossible, without making more or less complicated adjustments. The ease, however, with which the demand curve *for any one year* may be expressed in terms of absolute quantities[13] makes it possible to put the supply curve into juxtaposition with the proper demand curve.

Furthermore, an examination of the law of demand for successive years enables us to see the way in which the demand curve has shifted as a result of secular changes.

6. Finally, we should like to point out the real nature of the statistical law of demand. Some economists, among whom are to be included not a few members of the institutional school, have, unfortunately, gotten the impression that any attempt to derive a law of demand must needs be based on no better psychology than that of James Mill. A few of them even go so far as to deny the very existence of a law of demand. What these economists overlook, however, is that the existence of a law of demand is an objective fact, quite independent of one's psychological preconceptions. And when economists, in the words of Professor Mitchell, "grasp the idea that their business is with behavior, and that behavior is objective, they will see that their psychological footing can be made secure."[14] For the law of demand is not a fiction of the hedonistic school. It is nothing less than a summary presentation, in quantitative terms, of an important aspect of human behavior.

[13] Cf. pp. 40–42, 61–62, 74–75, and 85–87.

[14] W. C. Mitchell, "The Prospects of Economics," in R. G. Tugwell, *The Trend of Economics*, p. 23.

PART II. THE STATISTICAL LAW OF SUPPLY

CHAPTER IV

THEORETICAL CONSIDERATIONS RELATING TO SUPPLY

I. INTRODUCTION

"We should have made," says Marshall,[1] "a great advance if we could represent the normal demand price and supply price as functions both of the amount normally produced and of the time at which that amount became normal." In Part I we have derived the demand curve for sugar and have shown how it shifts its position from year to year as a result of dynamic changes. The curve thus derived may not be "normal" in Marshall's sense. It has, however, the great advantage of being concrete or statistical rather than a priori or hypothetical, and Marshall himself has warned us against the danger of abstract reasoning, especially in connection with supply and demand.

In Part II an attempt is made to treat supply along the same general lines as demand. More specifically, it is the object of the following chapters to examine some of the difficulties which arise in the derivation of the elasticity of supply; to indicate how these difficulties may be overcome; to derive the supply curve for sugar; to bring it, if possible, into juxtaposition with the demand curve for the same commodity, and to show how the equilibrium of demand and supply changes from time to time. In Part III an attempt is made to indicate the bearing of the results on the tariff on sugar. However, before proceeding to take up concretely the proposed problems, it is advisable to define a few terms and to examine some theoretical considerations relating to supply.

[1] *Principles of Economics* (8th ed.), p. 809.

II. THE CONCEPT OF THE ELASTICITY OF SUPPLY

The coefficient of the elasticity of supply might almost be defined by paraphrasing the definition of the coefficient of the elasticity of demand.[2] It is the ratio of the relative change in the quantity supplied to the corresponding relative change in price, when the relative changes are infinitesimal.

In mathematical symbols, the coefficient of the elasticity of supply, e, is

$$e=\frac{dx}{x}\Big/\frac{dy}{y}=\frac{d\log x}{d\log y}=\frac{dx}{dy}\cdot\frac{y}{x} \tag{17}$$

where x is the quantity supplied and y is the price per unit. Equation (17) is identical with equation (1), page 4, relating to η except that x now stands for the quantity supplied instead of demanded. If e is numerically greater than unity, the supply is said to be "elastic." If it is numerically less than unity, the supply is said to be "inelastic." The size of the coefficient measures the degree of the elasticity.

With regard to this definition the same observations must be made that were emphasized in connection with the definition of the elasticity of demand:

1. The definition presupposes a knowledge of the supply curve. It follows, therefore, that any attempt to derive the coefficient of the elasticity of supply of a commodity without first deriving the equation to the supply curve is apt to lead to difficulties.

2. The coefficient relates to a *point* on the supply curve and may vary in magnitude from point to point. In giving *the* coefficient of the elasticity of supply of a commodity, one must, therefore, specify the point on the supply curve to which it applies, unless the value of the coefficient is the same at every point on the curve.

This, of course, is equivalent to saying that the definition of the elasticity of supply given above is that of "point elasticity"

[2] See p. 4.

and not "arc elasticity," and the observations which were made on this distinction in connection with the discussion of demand[3] also hold for supply. We shall return to this subject later.

III. RELATION BETWEEN ELASTICITY OF DEMAND AND ELASTICITY OF SUPPLY

There is an interesting similarity between demand and supply in regard to elasticity. *In a closed economy the elasticity of supply is numerically equal to the elasticity of demand when the relative prices and the relative quantities are periodic functions of time.* By the terms "relative prices" and "relative quantities" are meant prices and quantities referred to some "normal" or trend as a base in order to allow for secular changes. By the term "closed economy" is meant an economy that is self-contained with respect to the commodity under consideration. Thus the United States is a closed economy with respect to potatoes. To get the corresponding economy for wheat or pig iron we must include the whole world.[4]

[3] See pp. 9–14.

[4] The importance of the "self-contained economy"—the one that has no relations of importance with anything outside of itself with respect to the commodity under consideration—has been stressed by Professor Cassel:

"First of all, science must always, in discussing causes and effects, take the *whole complex* as the object. It can not stop at any arbitrarily chosen link in the chain. It must consider the totality of occurrences which are in economic reality inextricably connected with one another. . . . .

"The practical consequence for economic science is that, in principle, it must always take an entire economic unit as its object. This means that economic science must always assume the economy which it wishes to analyse, as being enclosed within itself and having no connections with an outside world. For if any such connection should exist, it would be necessary, in order to get a complete view of the totality of the chains of causes and effects, to take even these connections into consideration, i. e., to widen the object of the investigation and regard a greater, but in itself complete economic unit. The unit chosen may be a small one, e. g., an isolated peasant economy, or a large one, the economy of a modern people or of the whole world. The object must be determined by the character of the investigations, but in any case it is essential that it should be a closed economy" (Gustav Cassel, *Fundamental Thoughts in Economics*, pp. 13–15).

See also his *Theory of Social Economy*, pp. 3–4.

In such an economy there are no imports or exports. Production is equal to consumption. The same data must, therefore, be used in deriving both the demand curve and the supply curve. Since the same quantity series must be correlated with the same price series both for demand and supply, it is clear that the first correlation must be negative and the other positive. Let us assume that a correlation of *synchronous* items yields the perfect result $r=-1.0$. (This will be obtained when the two synchronous series are in opposite phase, i.e., when the highest price corresponds to the lowest consumption.) The line of regression corresponding to this coefficient is the curve of demand. Since each of the two correlated series is, by assumption, a periodic function of time, it follows that a perfect *positive* correlation, $r=+1.0$, will be obtained by lagging production behind prices by half a period, and thereby bringing the two in the same phase. If such a correlation has economic meaning—*and this must be determined by reference to the technology of the industry or commodity under consideration*—the line of regression corresponding to this coefficient is the curve of supply. Under these conditions it can easily be shown[5] that the elasticity of supply is numerically equal to the elasticity of demand.

In any practical problem it is extremely unlikely that the quantity series and the price series would both be periodic functions of time and that the same quantity series would have to be used in the derivation of both the demand curve and the supply curve. From this it would be wrong to conclude, however, that the principle of the numerical equality of the elasticities of demand and supply is of no importance whatsoever. Where the necessary conditions obtain at least approximately, the elasticity of supply will *tend* to be numerically equal to the elasticity of demand as the number of observations increases.

A fuller explanation of the *statistical* reasoning which leads to this principle or theorem will be postponed until we are ready

[5] See pp. 132–36.

to derive the supply curve for sugar. At this point we may point out a practical application of the principle under consideration. Assuming that the world is the only self-contained economy with respect to sugar, and assuming also that the other necessary conditions are realized, suppose that we know the elasticity of the world-demand and the elasticity of the supply from all sources except Cuba. By our principle, which states that in a self-contained economy the elasticity of the whole demand tends to be numerically equal to the elasticity of the whole supply, the elasticity of the unknown Cuban supply may be easily estimated from the equation,

$$Q_w\eta_w = Q_c e_c + Q_o e_o \tag{18}$$

where η_w is the numerical value of the elasticity of the world demand, Q_w the total quantity demanded by the world's population; e_c, e_o are respectively the elasticities of the Cuban supply and the supply from all other sources; and Q_c, Q_o are respectively the quantities supplied by Cuba and the other sources, the relation between the quantities or weights being $Q_w = Q_c + Q_o$. By dividing both sides of this equation by Q_w, it will be seen that this equation simply states that the elasticity of the world-demand is a weighted arithmetic mean of the elasticities of the supplies from the various regions.

The principle of the equality of the elasticities of demand and supply is, however, true only of short-time supply and demand curves. As Marshall has pointed out in another connection, the matter is more complex in the more fundamental questions which relate to long periods. "For the ultimate output corresponding to an unconditional demand at even current prices would be theoretically infinite; and therefore the elasticity of supply of a commodity which conforms to the law of Increasing Return, or even to that of Constant Return, is theoretically infinite for long periods."[6]

[6] Marshall, *op. cit.*, pp. 456, 457.

IV. SUPPLY AND COST OF PRODUCTION

The cost curve gives the relation between the cost of production of a commodity and the quantity produced. The supply curve gives the relation between the price of the commodity and the quantity offered for sale or produced. This definition, although perfectly accurate, has, nevertheless, the shortcoming of not even suggesting the problem of the relation between the cost curve and the supply curve. Perhaps this is as it should be, for economists have not as yet succeded in giving an entirely satisfactory explanation of the relation between the two curves. If, however, we are permitted to make the convenient assumption —which is true under equilibrium conditions—that no part of the output is produced at a loss (i.e., that there is no ultra-marginal production), we may define the supply curve as the curve which gives the relation between the marginal cost, or the cost of the last-made portion of the commodity, and the quantity produced. This definition has at least these advantages: It suggests that there is a functional relation between the supply curve and the cost curve; and it leads to a mathematical expression for this relation under equilibrium conditions.

I. COST CURVES

The only condition of the cost curve is that the total cost of production of any quantity shall always be less than the total cost of production of a greater quantity, for otherwise the greater quantity would be made, and, if needful, partially destroyed. Mathematically this is expressed by saying that

$$\frac{dy}{dx}>0 \tag{19}$$

where $y=\phi(x)$ is the total cost of production (not the cost per unit) and x is the quantity produced.[7] There are three types of

[7] The verbal part of this definition of a cost curve is based on Henry Cunynghame's definition of a "supply" curve. (See his paper on "Some Improvements in Simple Geometrical Methods of Treating Exchange Value, Monopoly,

cost curves fulfilling condition (19). One of them is shown in Figure 24*A* (curve *OA*). It is the total cost curve (*Gesammtkostenkurve*) of Auspitz and Lieben.[8] The abscissa of any point on this curve represents the total output (production) during a specified period of time—say, one year—and the ordinate the *total cost*.

In Figure 24*B* (curve *sa*) there is shown the more common method of representing the cost curve. Here the abscissa represents the total production, as in curve *OA* (Fig. 24*A*); but the ordinate now represents the *average cost per unit*.

The relation between *sa* and *OA* is simple. If the former be denoted by $f(x)$ and the latter by $\phi(x)$, then

$$\left.\begin{aligned} \frac{\phi(x)}{x} &= f(x) \\ \text{or,} \qquad \phi(x) &= xf(x) \end{aligned}\right\} \qquad (20)$$

The theoretical cost curve as defined above must not be confused with the accounting cost curve which has been made familiar to economists through publications of the United States Tariff Commission, the Federal Trade Commission, and other Government bureaus, and which is known technically as Marshall's "particular expenses curve."[9] In the theoretical curve as

and Rent," *Economic Journal*, II [1892], 40.) The mathematical translation of the definition is, however, quite different from Cunynghame's, which is "that $\frac{dy}{dx}$ must always be greater than $\frac{y}{x}$." The difference between the two mathematical formulas is this: Whereas Cunynghame's formula excludes from the category of "total cost curves" all curves that are not concave upward (Figs. 27*C* and 27*E*), ours (equation [19]) includes them. Cunynghame's mathematical formula is not an accurate translation of his definition.

[8] *Recherches sur la Theorie du Prix* (Paris, 1914), pp. 2–5. Translated from the German *Untersuchungen über die Theorie des Preises* (1889).

By the cost of production of an article Auspitz and Lieben mean "the minimum amount of money which producers of this article, taken together, must receive in order to be able to produce it without loss" (pp. 2–3).

[9] Marshall, *op. cit.*, pp. 810–11.

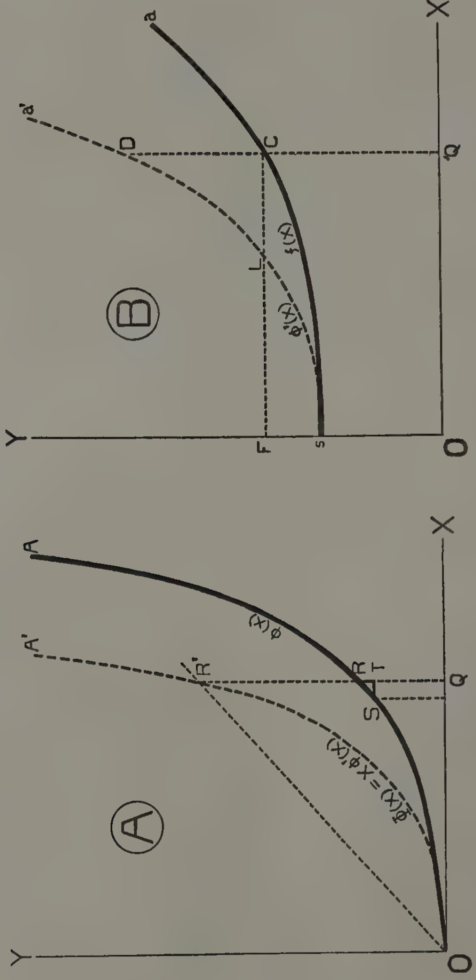

FIG. 24.—Relation between the theoretical cost curve and the theoretical supply curve under conditions of increasing cost.

A, total cost curve, $\phi(X)$, and total supply curve, $\Phi(X)$.

B, unit cost curve, $f(X)$, and unit supply curve, $\phi'(X)$.

conceived by Auspitz and Lieben, the abscissas represent quantities "which may be produced annually" by all the producers of the article under consideration. A change in the abscissa of the theoretical curve represents, therefore, a change in the *scale of production* from year to year. In the accounting cost curve the abscissas represent quantities which *were produced* by the various producers during the year or period in question. In this curve the scale of production is fixed and is represented by the base of the curve. This base is made up of the bases of a series of adjoining rectangles, arranged in ascending order, each rectangle representing by its base the output (and by its altitude the cost) of one producer during the year under consideration. The theoretical cost curve is expressed by a series of hypothetical or conditional statements thus: "If the annual (or other periodic) output of all the producers were so much, their total cost would be so much; if the annual output were so much again, the total cost would correspondingly vary, and so on." (This does not mean, of course, that the theoretical curve may not be based on inductive investigations.)

The accounting cost curve is expressed by a series of categorical statements thus: During the year (or period) covered by this cost survey, the producer who had the lowest average cost of production per unit had an output of so many units and a total cost of so many dollars; the producers whose average cost per unit was just above that of the lowest cost producer had an output of so many units again, at a cost of so many dollars per unit, making the total cost of the two producers so much, and so on.

Another important difference between the theoretical and the accounting cost curve is that the former does, while the latter does not, give the true relation between changes in cost and corresponding changes in output. The cost data which it has thus far been found possible or practicable to obtain show the *average* cost per unit and the output of each factory under consideration. They do not show the *variation* in the costs of each

factory as affected by the factory's output. The result is that the output corresponding to given unit (and hence, total) costs cannot be derived accurately from such data. Suppose that of several producers one has an output of 1,000 units at an average cost of $1.00 per unit and the other has an output of 2,000 units at an average cost of $2.00 per unit. This does not mean that all of the 1,000 units of the first producer cost $1.00 apiece to produce and that all of the 2,000 units of the second producer cost $2.00 apiece. Some of the 1,000 units may have cost more than $2.00 apiece and some of the 2,000 units may have cost less than $1.00 apiece. A true cost curve would show the relation between any given cost and the true cumulative output corresponding to this cost. The accounting cost curve, however, cannot show this, for it does not take into consideration the fact that the various units of a factory's output are produced at different costs.[10]

The economists who first made cost-of-production surveys usually organized their data in the form of ogives. They would have done better to have organized the same data in the form of frequency curves, for then the impression would not have arisen that the ogive is the same as the theoretical cost curve *oa* and that it may even be considered a supply curve. For most theoretical purposes, however, neither the frequency curve nor the ogive is as significant as the corresponding curve of total costs.

The relation between the three curves is illustrated numerically in Table V by means of an hypothetical example. In the same table there is also shown the relation between the curve of total costs and the curve of marginal costs (supply curve), which will be explained later. In this example the same data will do service both for the particular expenses curves and the theoretical cost curves.

[10] This point was also made by Professor Jacob Viner. In his review of Philip G. Wright's "Sugar in Relation to the Tariff" (*Journal of the American Statistical Association,* December, 1924, p. 545) there appears this statement: ". . . . the economist would ordinarily contend that there are different costs for different portions of the output of each producer, and that every producer tends to be a marginal producer with respect to a portion of his output."

The first two columns give the frequency distribution of costs. Thus, column (2) tells us that of the total output under consideration (100 tons), 10 tons were produced at an average cost of between \$0.50 and \$1.50 per ton; 24 tons were produced at an average cost of between \$1.50 and \$2.50 per ton; etc.

To represent the same cost data by an ogive, we simply cumulate the frequencies or output (col. [3]) and plot this cumulative output against the lower limits of the class intervals of column (1). The cumulative frequencies are plotted on the horizontal axis, and the corresponding lower limits of the unit costs

TABLE V

NUMERICAL ILLUSTRATION OF THE RELATION BETWEEN THE FREQUENCY DISTRIBUTION OF COSTS, THE OGIVE COST CURVE, THE CURVE OF TOTAL COSTS, AND THE CURVE OF MARGINAL COSTS

Average Cost per Ton (Dollars)	Frequency, or Output Produced at Specified Cost (Tons)	Cumulative Output* (Tons)	Cumulative Total Costs (Dollars)	Cumulative Marginal Costs (Receipts) (Dollars)	Unit Costs Derived From	
					Total Costs (4)÷(3) (Dollars)	Marginal Costs (5)÷(3) (Dollars)
(1)	(2)	(3)	(4)	(5)	(6)	(7)
0.50–1.50......	10	0	0	0	0.00	0.0
1.50–2.50......	24	10	10	10	1.00	1.0
2.50–3.50......	32	34	58	68	1.71	2.0
3.50–4.50......	24	66	154	198	2.33	3.0
4.50–5.50......	10	90	250	360	2.78	4.0
		100	300	500	3.00	5.0
	100					

* I.e., output having costs less than the lower limits of the successive figures of column (1).

In any practical problem such errors in quadrature as may be introduced in the process of cumulation ought to be allowed for, especially in the computation of the figures in the remaining columns.

on the vertical axis. Sometimes the variables are interchanged; but the method described above is generally used when the data relate to costs, since it is often (erroneously) assumed that the ogive is a true supply curve, and we know that the latter is represented by plotting output as abscissas and prices as ordinates.

Figure 25, *A* and *B*, is a graphic representation of the relation between the ogive and the frequency curve of costs.[11] In constructing Figure 25 the assumption has been made that the total output is fixed at 100 tons and that the first two columns of Table V simply give the costs of the five fractional parts of

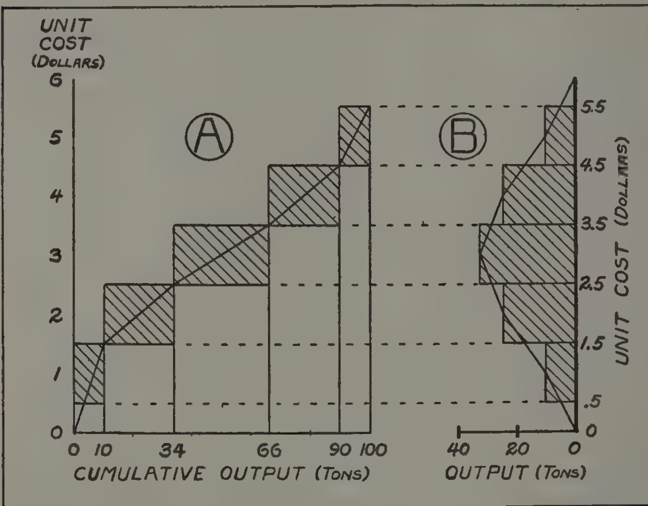

FIG. 25.—Relation between the accounting cost curve, or ogive, and the frequency curve of costs.

A, Ogive of costs, cumulated upward.

B, Frequency distribution of costs.

this output. It is convenient to think of this output as being produced by five different factories (or farms). From this point of view the ogive (Fig. 25*A*) is a series of rectangles, arranged in ascending order of height, each rectangle representing by its base the production of a factory (or farm) and by its altitude

[11] For a fuller description of the relation between the ogive and the frequency curve see Francis Galton, *Natural Inheritance,* p. 38, or Robert E. Chaddock, *Principles and Methods of Statistics,* p. 63.

the highest unit cost of the same factory. It is, in other words, an accounting cost curve of the type made familiar to economists through the publications of the War Industries Board, the Tariff Commission, the Federal Trade Commission, and other bodies. It is not a theoretical cost curve.

To use the same data for illustrating the construction of the theoretical total cost curve, we must drop the assumption of a fixed output of 100 tons and think of the figures in column (3) (cumulative output) as representing *hypothetical annual outputs of the industry as a whole.* These figures are plotted as abscissas and the corresponding *total costs* as ordinates. The latter are obtained as follows: When nothing is produced, there are no costs to be incurred (by assumption); the curve therefore begins at the point 0,0, or the intersection of the co-ordinates. The ordinate of this point is represented by the first figure (0) of column (4). When 10 units are produced the total cost is 10 times \$1.00 (the mid-value of the first cost interval), or \$10.00. This is represented by the second figure in column (4). When 34 units are produced (10+24), the cost of the first 10 units is, as we have just seen, \$10.00; and the cost of the next 24 units is 24× \$2.00=\$48.00 (see cols. [1] and [2]); the cumulative cost of the 34 units is, therefore, \$10.00+\$48.00=\$58.00. This is represented by the third figure in column (4). Proceeding in this manner, we obtain the rest of the figures in column (4).

Figure 26*A* shows the total cost curve thus obtained (curve *OA*). It corresponds to curve *OA* in Figure 24*A*.

If we divide the "cumulative total costs" (col. [4]) by the cumulative output (col. [3]), we obtain the average unit costs shown in column (6). When these average unit costs are plotted as ordinates corresponding to the cumulative output as abscissas, we obtain curve *oa,* Figure 26*B*, which corresponds to curve *sa,* Figure 24*B*. Its relation to curve *OA* is given by equation (20). (The meaning of curves *OA′* and *oa′* will be explained later.)

It will be seen by comparing curve *sa* (Fig. 24*B*) with curve

oa (Fig. 26*B*) that the former is concave upward while the latter is concave downward. There is no inconsistency in this. The fact that the total costs of a commodity increase at an increasing rate as production increases does not necessarily mean that the average cost per unit must also increase at an increasing rate; the average unit cost may increase at a *decreasing* rate.

It will also be observed that curve *oa* begins at the origin, while curve *sa* begins at a point (*s*) on the vertical axis, above

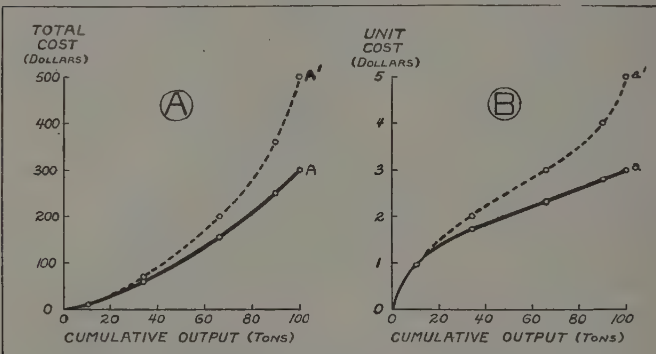

FIG. 26.—Derivation of the supply curve from the cost curve.
A, Total cost curve, *OA*, and total supply curve, *OA'*.
B, Unit cost curve, *oa*, and unit supply curve (curve of marginal unit costs), *oa'*.

the origin. There is nothing wrong about this, either. The beginning of the unit cost curve is determined by the slope of the total cost curve *OA* at its origin, and the slope may have a positive value even at the origin.

Curve *sa* (or its equivalent, *oa*) must not be confused with the ogive. The latter, as has already been pointed out, may be conceived of as a series of (small) rectangles, arranged in ascending order of height, each rectangle representing by its base the production of a factory (or farm) and by its altitude either the highest or the average *cost per unit of the same factory.*

Curve *sa* may also be looked upon as a series of (small) rectangles arranged in ascending order of height and having the same bases as in the ogive, but the altitudes of the rectangles now measure a different variable. They represent *the average cost per unit of all factories whose unit costs do not exceed that of the one under consideration.* In other words, the altitude of each rectangle measures the average (weighted) of the altitudes of all the rectangles to the left of it.

If we had detailed cost data for each factory for a series of years, we would probably find some factories, if not some industries, in which increases in production from one year to the next are accompanied by *decreases* in unit costs—at least within certain limits. Curve *sa* would then be inclined negatively. Few inductive investigations, however, result in such data as would throw light on the most important question of increasing and decreasing costs, partly because most investigations are not conducted with the view of throwing light on economic theory and filling some of the "empty boxes" and partly because such data are very difficult to obtain. All that the ordinary cost of production survey succeeds in showing are the output and the average cost per unit of each factory for the year under consideration. Theoretically there is no more reason for arranging such data according to the ascending order of unit costs and thus obtain an ogive that is cumulated upward than for arranging them according to the descending order of unit costs and thus obtain an ogive that is cumulated downward. Just as the former does not suggest increasing costs so does the latter not suggest decreasing costs. To obtain some light on the problem of increasing or decreasing costs, we must find the relation between changes in the *scale* of production from one year to the next and corresponding changes in costs.

The general presumption, however, is against the existence of decreasing cost industries in a régime of free competition, if by an industry we understand, with Professor Frank H. Knight,

the entire process (all stages) of producing the article in question.[12]

2. SUPPLY CURVES

Returning now to the theoretical curve of total costs (OA, Fig. 24A) we recall that the abscissa of any point on the curve represents the quantities which can be produced and sold annually and the ordinate the total cost of production. This amount, however, is not what producers, taken collectively, will actually receive, for they will try to obtain a price which is over and above the cost of production. How can the amount actually received by the producers be determined? What is the relation between cost of production and price?

Let OQ (Fig. 24A) be the equilibrium production for any one year, and ST the last or marginal portion produced. Then QR is the total cost of the entire output OQ, and TR is the cost of the marginal portion. Under free competition the marginal cost per unit is equal to the price, or the price is $TR:ST$. But in a freė market each unit of the produce must be sold at the same price as any other unit, so that the total annual output OQ will be sold at exactly the same price as the last portion, ST. This means that the sum of money Y' received for OQ must obey the relation

$$Y' : OQ :: TR : ST$$

or

$$Y' = \frac{TR}{ST} \times OQ = QR' ,$$

which is greater than QR.

Now let the equilibrium production be any other value than OQ. By the same reasoning we compute a new value for Y' corresponding to the new equilibrium.

[12] "Any branch or stage in the creation of a product which offer continuously a chance for technical economies with increases in the scale of operations must eventuate either in monopoly or in leaving the tendency behind and establishing the normal relation of increasing cost with increasing size" ("Some Fallacies in the Interpretation of Social Cost," *Quarterly Journal of Economics*, XXXVIII [1924], 597).

Proceeding in this manner, we obtain a series of points which, when joined together, give the new curve OA'. This curve is above OA if the latter is concave upward.

The same curve may be obtained by a slightly different method, as follows: Through the origin O draw a line parallel to the tangent of the total cost curve at the point R. This line crosses the prolongation of the ordinate of the point R at R'. The length QR' gives, as before, the total amount of money which must be paid to all producers if they are to produce the quantity OQ, for $QR':OQ::TR:ST$. Make the same construction for all other points of tangency on curve OA, and you obtain the new curve OA', as before. This, in fact, is the method followed by Auspitz and Lieben.

Each ordinate of the curve OA' gives the actual amount which must be paid to all the producers if they are to be induced to produce, during the year, the quantity of the article represented by the corresponding abscissa. Or better still, the ordinates of this curve indicate the amounts of money against which are offered, in fact, the quantities annually produced and designated by the abscissas. Accordingly, Auspitz and Lieben have named this curve the total or collective supply curve (*Gesammtangebotskurve*).[18]

Mathematically, the relation between the total supply curve OA' and the total cost curve OA is

$$\Phi(x) = x\phi'(x) \tag{21}$$

where $\Phi(x)$ is the total supply curve and $\phi(x)$ is the total cost curve.

If we divide the ordinates of the curve OA' by their corresponding abscissas we obtain a new curve $sa' = \phi'(x)$ (Fig. 24*B*), or

$$\frac{\Phi(x)}{x} = \phi'(x) \tag{22}$$

[18] Auspitz and Lieben, *op. cit.*, p. 8.

which is related to OA' in the same way as sa is related to OA. (See equation [20]). This curve, which has been called by Professor Pigou "the curve of marginal supply prices"[14] and by Professor Young "the curve of aggregate expenses,"[15] we shall call simply "the supply curve," in order to show its relation to the Auspitz and Lieben "total supply curve" (*Gesammtangebotskurve*), OA'.

This curve is thus related to sa. If the ordinate at any point of the abscissa, Q, (Fig. 24, B), intersects sa at C and sa' at D, the area $OQDs$ is equal to the area $OQCF$, or, in symbols,

$$\int_0^x \phi'(x)dx = xf(x) = \phi(x) \, . \tag{23}$$

But the area $OQCF = xf(x) = QR$ (Fig. 24, A) represents the total cost of production of the quantity OQ. Accordingly the line QD—or, more exactly, the little rectangle of which the line is the height and a small unit of produce is the base—represents *"the addition to the total cost incident to the production of an additional unit."*[16]

But, it will be asked, "Is not this the very definition of QC? Your mathematics to the contrary, notwithstanding, how can the increment in cost be QD?" The answer is that the two lines represent different concepts of marginal costs. The former is the increment in cost consequent on an increment in production

[14] A. C. Pigou, *The Economics of Welfare* (1920), p. 931. See also Professor Pigou's article on "Producers' and Consumers' Surplus," *Economic Journal*, XX, 358–70.

[15] A. A. Young, "Pigou's Wealth and Welfare," *Quarterly Journal of Economics*, August, 1913, pp. 676–77.

Professor Young seems to prefer this term because the *area* inclosed by this curve (sa', Fig. 24, B), the two axes, and the ordinate at any point "represents the aggregate expenses" of producing OQ units. But it is more common to designate a curve after the variable measured by its *ordinates*. From this point of view it would be better to confine the term "curve of aggregate expenses" to OA' (Fig. 24, A) and to call sa' by some other term, if the one used by Pigou is open to objection.

[16] F. Y. Edgeworth, *Papers Relating to Political Economy*, II, 433.

when the prices of the factors of production are given. The latter is the increment in cost consequent on an increment in production, when the prices of the factors of production are *not* given, but vary with changes in the scale of production. That is, QC is the marginal unit cost from the point of view of the entrepreneur whose production is too small as compared with the aggregate output to affect the prices of the factors of production. But QD is the marginal unit cost *viewed socially*.[17]

Reference to Table V and Figures 26*A* and 26*B* will make this clear. When the output is 10 tons (col. [3]) the average cost is $1.00 per ton (i.e., the midpoint of the first cost interval of column [1]), and the total cost is $10.00 (col. [5]). This is a point common to both curve OA and curve OA' (Fig. 26*A*). When 24 *more* tons are needed, these cannot be produced cheaper than at $2.00 per ton (i.e., the midpoint of the second cost interval of col. [1]). This is (approximately) the cost of the last unit which is equal to the price. If it be assumed that the higher unit cost of the second dose will not affect the unit cost of the first 10 tons, the total cost would be (10×$1.00) + (24×$2.00)=$58.00 (col. [4]). This is a point on curve OA (Fig. 26*A*). But under free competition the price of the factors of production—including, of course, the rent of land—will be bid up when the price rises to $2.00 per ton, so that even the first dose of 10 tons will now cost $1.00 per ton *more* than formerly. That is, *viewed socially*, the increased cost of obtaining the additional 24 tons is not merely $48.00 but $58.00, or the cumulative (marginal) cost of the 34 (10+24) tons is $68.00 (col. [5]). This is a point on curve OA' which is now above curve OA. Proceeding in this manner, we obtain the other figures of column (5), or the other points of curve OA'. The difference between $68.00 and $58.00, or, more generally, the difference between any figure in column (5) and the corresponding figure in column (4), corresponds to the difference between QR' and QR (Fig. 24*A*).

[17] This interpretation differs from that given by Edgeworth. See *ibid.*

Dividing the cumulative marginal costs (col. [5]) by the cumulative output (col. [3]), we obtain the marginal *unit* costs of column (7). Curve oa' (Fig. 26*B*), represents these unit costs. It is the ordinary supply curve. It corresponds to curve oa' (Fig. 24*B*). Its relation to curve oa is given by equation (23).

3. LAWS OF COST

In the foregoing analysis of the relation between the cost curve and the supply curve, the total cost curve was drawn concave upward, and it was shown that when the cost curve is of this type the supply curve is *above* the cost curve. Although this is a common type of cost curve, it is by no means the only one fulfilling the general condition of the cost curve (see equation [19]) that the total cost of production of any quantity shall always be less than the total cost of production of a greater quantity, for otherwise the greater quantity will be produced, and, if needful, partially destroyed. There are two other cost curves which fulfil this condition. One is concave downward, and the other has no curvature, being a straight line. The three curves, together with the unit cost curves derived from them are shown in Figure 27. Of these, the two curves *OA* of Figures *A* and *C* may also be looked upon as two arcs of one continuous total cost curve which is concave downward for part of its length and concave upward for the rest of its length. A similar statement may be made of the unit cost curves *oa* of Figures B and D. It can easily be shown that when the cost curve is concave downward, the curve of marginal costs is *below* the cost curve, and that when the cost curve has no curvature, the curve of marginal costs coincides with the cost curve.[18]

But whether the curve of marginal costs is above or below the cost curve, it is clear that any dealer to sell at all, must ex-

[18] By equation (21) the curve of marginal costs $\Phi(x)$ is thus related to the cost curve $\phi(x)$. $\Phi(x)=x\phi'(x)$. In Figure 27, *A* and *B*, $\phi'(x)$ is an increasing function of x as x increases, hence $\Phi(x)$ is *above* $\phi(x)$. In Figures *C* and *D*, $\phi'(x)$ is a decreasing function of $\phi(x)$ as x increases, hence $\Phi(x)$ is *below* $\phi(x)$. In Figures *E* and *F*, $\phi'(x)$ is a (positive) constant, hence $\Phi(x)=\phi(x)$.

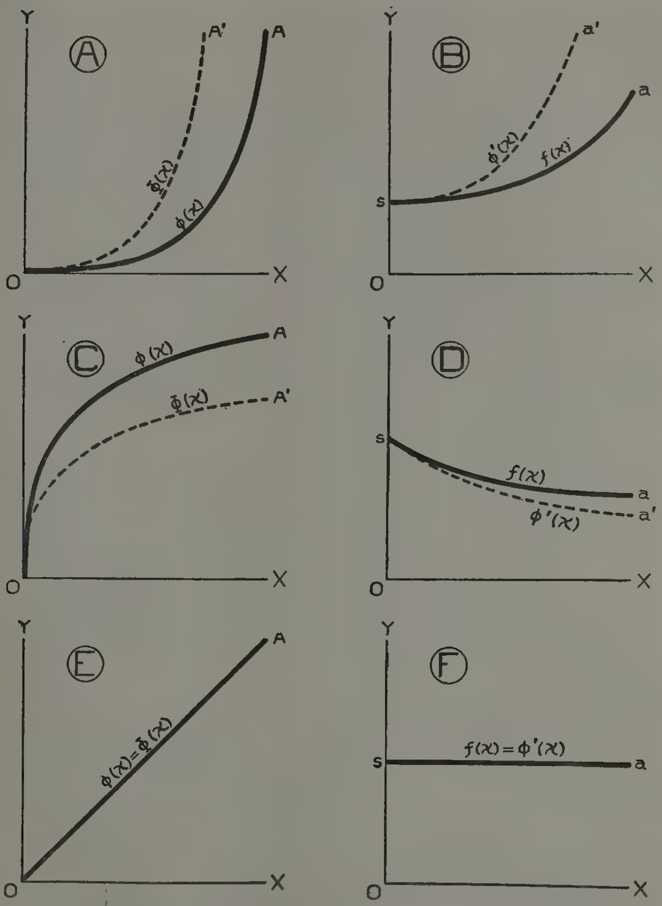

FIG. 27.—The three laws of cost.

Figures *A*, *C*, and *E* represent the three kinds of *total* costs, while *B*, *D*, and *F* represent the corresponding *unit* costs. The dotted curves represent *marginal* (total and unit) costs.

pect to get back at least the total cost. As Professor Fisher puts it,

> This means that he must, therefore, charge a price *at least* as high as the *average* cost per ton. When the cost of each successive ton is *greater* than that of the preceding ton, the cost of the last, the marginal cost, is the greatest cost of all, and therefore exceeds the average cost. Consequently, the dealer is assured a profit when selling at a price equal to the marginal cost. But when the cost of each successive ton is *less* than that of the preceding ton, the cost of the last ton (marginal cost) is the least of all, and therefore is less than the average cost. To sell at a price equal to marginal cost would, in this case, mean to sell at a loss.
>
> In either case, whether the curve ascends or descends, the seller will seek to determine his price on the basis of the higher of the two costs (marginal and average). Whichever of the two is the higher will be the one to appear in the supply curve. When the marginal cost increases with supply, marginal cost is the higher, and will rule supply. When the opposite is true, average cost is the higher, and will rule supply.[19]

But if the cost curve descends to the right, it can easily be shown[20] that such a condition will lead to cutthroat competition and ultimately to monopoly. A descending cost curve is, therefore, incompatible with the assumption of free competition. Under free competition the supply curve is above the cost curve.

It is clear, therefore, from the foregoing, that, given the cost curve, we can find the corresponding supply curve; and vice versa.

The shape of the total cost curve not only determines the shape of the corresponding supply curve but it also enables us to state the laws of cost in an unequivocal manner. This will be illustrated by a consideration of the laws of cost as conceived

[19] Irving Fisher, *Elementary Principles of Economics* (1913), p. 316. See also Luigi Amoroso, *Lezioni di Economia Matematica* (1921), pp. 167–87, where he distinguishes between *costo unitario, costo marginale,* and *costo virtuale.*

[20] Fisher, *op. cit.*, pp. 317–23; see also note 12, above.

It was Cournot, however, who first pointed out that a negatively-sloping cost curve is incompatible with the assumption of free competition. See his *Researches into the Mathematical Principles of the Theory of Wealth* (Bacon's translation), pp. 91–92.

by two of the greatest mathematical economists, Cournot and Edgeworth.

Cournot, who was the first to give a description of the laws of cost or return in mathematical form,[21] shows that if $y=\phi(x)$ is the equation to the total cost curve, then there are three types of laws of cost according as

$$\phi''(x) \gtreqless 0 . \tag{24}$$

Of this statement of the laws of cost Professor Moore says:

> He [Cournot] avoided many difficulties by contenting himself with a mathematical definition of the laws without passing on to identify them by name as the law of diminishing return, the law of constant return, and the law of increasing return. Without using the customary unprecise definitions he amply made good his claim that the condition whether $\phi''(x) \gtreqless 0$ exerts very great influence on the principal problems of economic science. *It would be conducive to clearness and accuracy if the Cournot criterion $\phi''(x)$ were regarded as the criterion of laws of cost or laws of return.*[22] [Italics ours.]

The late Professor Edgeworth gives several definitions of the laws of cost, most of them being basically translations in words of the Cournot criterion (24).

His *provisional definition* may be stated as follows: When on the production of two successive equal doses of produce the increment of cost due to the first dose is less than the additional increment due to the second, the law of increasing costs is said to act; and conversely it is a case of diminishing costs when the increment of cost due to the first dose is greater than the increment due to the second.[23]

[21] *Researches into the Mathematical Principles of the Theory of Wealth* (Bacon's translation), pp. 58–61.

In addition to these three types, Cournot considers a fourth, where the total cost curve is a horizontal line, i.e., where $\phi(x)$ is a constant.

[22] "A Moving Equilibrium of Demand and Supply," *Quarterly Journal of Economics*, XXXIX (May, 1925), 360, 361.

[23] F. Y. Edgeworth, *Papers Relating to Political Economy*, I, 61–69, paper on "The Laws of Increasing and Diminishing Returns."

For numerical illustration of this definition, based on Professor T. N. Carver's example, see p. 63.

Most of Professor Edgeworth's examples and definitions relate to the laws

In this definition the two successive doses of produce are assumed to be equal. But now, removing this restriction, we may, by paraphrasing Professor Edgeworth's *general definition* of the law of return, thus state the laws of cost: When on the production of two (not in general equal) doses of produce the increment of cost due to the two doses has to the increment of cost due to the first dose alone a ratio greater than the sum of the two doses to the first dose, the law of increasing costs is said to act; and conversely if the former ratio is less than the latter, the law of decreasing costs is said to act.[24]

The foregoing general definition comprises the particular and limiting case when the first of the two successive doses of produce is the whole production measured from zero; the second dose being larger or smaller according to the purpose in hand.[25] Thus understood, the definition comes to this, that the law of increasing costs acts when the *average* cost per unit of product increases with the increase of production (by an amount that is of an assigned order of magnitude); and the law of diminishing cost, in the converse case.

---

of *return*, rather than to the laws of *cost*. His producing doses and the resulting product are, therefore, mostly expressed in *kind*. "But no essential difference in classification is introduced when we take money as the measure; provided that the prices, both of the product and the factor of production, remain constant while the amounts are varied. For the change thus introduced is simply to multiply the axes representing outlay and return, ... each by a constant. But such a change does not alter the character of a curve in respect to convexity or concavity. . . . .

"But money can no longer be ignored when we consider price as varying with the amount put on the market by the individual entrepreneur. . . . ." (*op. cit.*, pp. 68–69).

For this reason it seemed best in this chapter to confine our comments to costs expressed in money rather than to returns expressed in kind.

[24] Let $y=\phi(x)$ be the cost curve. At any point x_1 increasing or decreasing costs are said to act according as

$$\frac{\phi(x_2)-\phi(x_0)}{\phi(x_1)-\phi(x_0)} > \text{ or } < \frac{x_2-x_0}{x_1-x_0}$$

where $x_0<x_1<x_2$.

[25] Putting $x_0=o$ in note 24, above.

This definition Edgeworth distinguishes as *secondary;* the general definition being called *primary*. Edgeworth prefers the primary definition,[26] although many other high authorities quoted by him give precedence to the secondary definition.

The primary and the secondary definitions do not necessarily lead to the same conclusion as to the kind of law of cost that may be operating in a given industry. For example, if the total cost curve OA (Fig. 24A) did not begin at the origin O but at a point above the origin, thus indicating an industry in which an initial outlay is required before any production could take place, it would have a tract—that extending from the new origin to the point where a tangent from O touches the curve—over which increasing costs would prevail according to the primary definition and decreasing costs according to the secondary definition. Beyond the point where the tangent touches the curve, increasing costs would prevail in both senses.

It will be observed that the statements of the laws of cost as given both by Cournot and Edgeworth are simply descriptions of the various shapes which the cost curve may assume. There is, however, an important difference between the definitions of the two authorities, which recalls the difference between "point" elasticity and "arc" elasticity of demand.[27] Cournot assumes for convenience that output and cost can move by infinitesimal steps, and that the function connecting output and cost is a continuous function. This assumption, we shall find, is also convenient in the mathematical treatment of concrete, statistical laws of supply. Edgeworth does not necessarily make this assumption. The doses that he considers move by finite steps. This makes it incumbent upon him to specify the magnitude of the doses under consideration.

In his own words:

> In order to make our definition precise it is often necessary to specify the *magnitude* of the doses successively applied. Otherwise it may happen

[26] For his grounds see *Papers Relating to Political Economy*, pp. 70 ff.

[27] See pp. 9–14.

that *both* Increasing and Diminishing Returns may truly be predicted of the same circumstances. This is a paradox familiar to those who are conversant with the application of the differential calculus. It depends on the circumstance that the orders of magnitude which may be neglected are different according to the different purposes contemplated. It is thus—to use Clerk-Maxwell's illustration—that the heterogeneities in the structure of a mound of gravel, which are negligible from the point of view of the military contractor, may be all-important to the worm. For a like reason the surface of a mountain at any assigned point, say that which is exactly underneath the center of gravity of an ascending mountaineer, may appear to him, while he surmounts the rough surface with long strides, to be shaped like a dumpling, concave with respect to the plane of the horizon; but to the beetle creeping up a cup-shaped cavity, convex.[28]

4. LAWS OF RELATIVE COST

In the preceding paragraphs the relation between cost of production and output has been expressed in terms of absolute changes in the total cost and the corresponding absolute changes in output. We shall get a deeper insight into this relation if we also consider the relation between relative changes in total cost and the corresponding relative changes in output.

"Relative cost of production" has been defined by Professor Moore as "the ratio of the relative change of the total cost to the relative change in the total production."[29] If $y=\phi(x)$ is the total cost of production and x is the total amount of production, the symbolic representation of the "coefficient of relative cost of production" is $\kappa=\frac{\Delta y}{y}\Big/\frac{\Delta x}{x}$, or at the limit,

$$\kappa=\frac{dy}{y}\Big/\frac{dx}{x}=\frac{x}{y}\frac{dy}{dx}=\frac{x\phi'(x)}{y}. \tag{25}$$

This criterion gives the information desired by the entrepreneur. He wishes to know, if he increase his output, whether the relative increase in

[28] *Papers Relating to Political Economy*, I, p. 65.

[29] "A Moving Equilibrium of Demand and Supply," *Quarterly Journal of Economics*, XXXIX (May, 1925), p. 359.

total cost will be greater than, equal to, or less than the relative increase in the output. That is, he wishes to know whether

$$\kappa = \frac{x\phi'(x)}{y} \gtreqless 1 \,.$$

Relative efficiency of organization, ω, gives, under a different form, the same information as the relative cost of production, κ. The criterion ω is defined as the ratio of the relative change in total production to the relative change in total cost. Symbolically,

$$\omega = \frac{dx}{x} \bigg/ \frac{dy}{y} = \frac{y}{x\phi'(x)} \,.$$

The information desired by the entrepreneur is whether $\omega \gtreqless 1$.

Since $\kappa = 1/\omega$, it is obvious that if the value of either κ or ω is known, the other may be found.[30]

Accordingly as $\kappa \gtreqless 1$ we have to do with the law of increasing relative cost, the law of constant relative cost, the law of decreasing relative cost. "Just as Cournot's criterion $\phi''(x)$ is the criterion of laws of cost or of return, so the coefficient may be regarded as the criterion of laws of relative cost or of relative return."[31]

Professor Moore shows, however, that the law of cost is a broader conception than the law of relative cost. He concludes his examination of these concepts as follows: "The criterion $\phi''(x)$ is more inclusive than the criterion κ: where κ occurs in a particular form the corresponding form $\phi''(x)$ always occurs, but when $\phi''(x)$ exists in any particular form the corresponding condition of κ may or may not be fulfilled."[32]

V. ELASTICITY OF SUPPLY AND RELATIVE COST OF PRODUCTION

And now we come to a difficult question: What is the relation between the elasticity of supply and the relative cost of production? In answering this question it will be somewhat more convenient to work with the reciprocal of the relative cost of production, or the relative efficiency of organization.

[30] *Ibid.*, pp. 359–60.

[31] *Ibid.*, p. 361.

[32] *Ibid.*, p. 364.

By definition, the coefficient of the elasticity of supply (e) is the ratio of the relative change in the quantity supplied to the corresponding relative change in price (page 100), and the coefficient of the relative efficiency of organization (ω) is the ratio of the relative change in total production to the relative change in total cost. We may assume for the purpose in view that "the quantity supplied" is equal to "the total production." The relation between e and ω must, therefore, be sought in the relation of price to the relative change in total cost.

We have seen, however (pp. 114–18), that the bridge by which we cross from cost to price and back to cost is *marginal* cost, for at the margin of production cost is equal to price. This brings us for the second time to a consideration of the nature of the curve of marginal costs.

In a previous section (pp. 114–21) we have defined the supply curve, or "the curve of marginal costs" as the rate of change of the total cost curve, or as $\phi'(x)$, the total cost curve being designated by $\phi(x)$. This is curve sa' (Fig. 24B). The definition of the elasticity of supply applied to this function ($\phi'(x)=sa'$) is

$$e=\frac{1}{\phi''(x)}\cdot\frac{\phi'(x)}{x}. \tag{26}$$

But a good many, if not most, economists also consider the curve of *average* unit costs, $f(x)$, or curve sa (Fig. 24B), as "the supply curve" even under conditions of increasing cost. This curve is simply the total cost of production divided by the total production, or $f(x)=\dfrac{\phi(x)}{x}$. The coefficient of the elasticity of supply derived from this function $\left(f(x)=\dfrac{\phi(x)}{x}\right)$ is

$$e=\frac{\phi(x)}{x\phi'(x)-\phi(x)}=\frac{\omega}{1-\omega} \tag{27}$$

where $\omega=1/\kappa$ is, by definition, $\dfrac{\phi(x)}{x\phi'(x)}$.

But what is the meaning of these different definitions of elasticity of supply? The answer must be sought in the distinction that we have drawn (see pp. 116–17) between two concepts of marginal costs. There is first the function giving the relation between an increase in cost and the corresponding increase in production when the prices of the factors of production are given. It is the function which represents marginal cost from the point of view of the entrepreneur whose production is too small compared with the aggregate output to affect the prices of the factors of production. The definition of elasticity of supply proper to this concept of marginal cost is equation (27). There is second the function giving the relation between an increase in cost and the corresponding increase in production when the prices of the factors of production are not given, but vary with the changes in the scale of production. It is the function which represents marginal cost from the point of view of society as a whole. The definition of elasticity of supply proper to this concept of marginal costs is equation (26).

VI. TYPICAL EQUATIONS TO THE LAW OF SUPPLY

From the relation existing between cost of production and supply, which has been briefly outlined in this chapter, we can deduce the supply curve from the corresponding cost curve and conversely.

For example, suppose that we have data giving the total costs corresponding to given units of output of a commodity and we desire to find the supply curve corresponding to these data. Since, under conditions of increasing cost, the supply curve is simply the rate of change, or the first derivative, $\phi'(x)$, of the total cost curve, $\phi(x)$, all that we have to do is to fit a curve $y=\phi(x)$ to the total costs and then differentiate this function with respect to x (output). The derived function is the equation to the supply curve.

If it be impracticable to fit a curve to the data of total costs, we may work with the finite differences of the total costs instead

of with the derivatives of the function $\phi(x)$. The series of the first differences of the total costs, properly smoothed, may give a fair approximation to the supply curve.

In nearly all inductive investigations, however, it will be impossible to obtain directly the equation to the total cost curve. It will, therefore, be necessary first to derive the supply curve and from it deduce the corresponding cost curve.

This makes it desirable, in the words of Professor Moore, "to select a typical equation to the law of supply so that the constants in the equation shall reflect the connection of supply with the laws of cost."[33]

Professor Moore has shown[34] how the types of the supply equations obviously vary according as the values of the Cournot criterion $\phi''(x)$ or of his (Moore's) own criterion κ are taken as the criteria of the laws of cost.

We shall first derive the equations of supply by means of $\phi''(x)$. The simplest possible assumptions as to the character of $\phi''(x)$ are summarized in (vii) [the (28) of this text]

$$\left.\begin{array}{l}\phi''(x)=a\\ \phi''(x)=a+bx\\ \phi''(x)=a+bx+cx^2\end{array}\right\}. \tag{28}$$

If $\phi''(x)=a$, then $\phi'(x)=ax+b$, and the variation of marginal cost is described by a straight line. When the law of diminishing return dominates industry, $\phi'(x)=ax+b$ is the equation to the supply curve. In that case, if p_s be put for the supply price per unit of commodity, the supply equation is $p_s=ax+b$. When the law of constant return dominates the industry, $a=o$, $\phi'(x)=$ a constant, and the supply price is $p_s=\dfrac{\phi(x)}{x}$. When the law of increasing return prevails, the supply price[35] is likewise $p_s=\dfrac{\phi(x)}{x}$.

More complex equations could be derived in a similar manner from the other assumptions in (28). But it is perhaps advisable, in any concrete investigation, to let the data suggest the most desirable equation for the purpose in view.

[33] *Op. cit.*, p. 364.
[34] *Op. cit.*, pp. 364–65.
[35] Marshall, *Principles of Economics*, 4th edition, p. 539, note 1. Fig. 36.

The criterion κ, Professor Moore also shows,[36] leads to another useful type of supply curves which is related in form to his "typical equation to the law of demand," to which reference was already made on page 56.

VII. SUMMARY

At the outset of this chapter we defined the coefficient of elasticity of supply and developed an interesting relation between the elasticity of supply and elasticity of demand. Proceeding to an analysis of the relation between supply and cost of production, we pointed out the general condition of the cost curve, and the difference between the theoretical cost curve and the accounting cost curve, with which it is frequently confused, and between the cost curve and supply curve. Continuing our analysis, we noted that the shape of the total cost curve not only determines the shape of the corresponding supply curve but also enables us to state the laws of cost and the laws of relative cost in an unequivocal manner, and we derived the equations connecting relative cost of production and elasticity of supply. Finally, we noted that the simple functional relation between the cost of production curve and the supply curve makes it desirable to select a typical equation to the law of supply whose parameters shall reflect the connection of supply with the laws of cost, and we described briefly Professor Moore's system of typical equations to the law of supply.

But the supply curves which have been treated thus far are purely theoretical supply curves. Although they should throw important light on concrete inductive supply curves, they are not themselves based on the facts of any industry and they lack, therefore, the utility and interest which would attach to the concrete, statistical supply curves. How can statistical supply curves be derived? How, for example, can the relation between changes in the price and changes in the output of sugar be determined quantitatively? This is our next major problem.

[36] *Op. cit.*, pp. 365–67.

CHAPTER V

THE LAW OF SUPPLY FOR SUGAR

I. THE RELATION OF THEORY TO METHOD

How, then, do we proceed to derive the law of supply? When one first begins to consider the many difficulties which beset this task—many more than were encountered in the problem of demand[1]—he is apt to conclude that it is impossible to derive the law of supply statistically. However, the situation becomes a little more hopeful and the inquiry is given a certain unity of purpose if we recall that the main object of the law of supply is to answer the question: "What must be the change in price for a given year or period to *call forth* a given change in the supply?"

But the change in supply may occur almost immediately after the change in price or only after the lapse of a considerable period of time, depending on the "period of production" of the commodity in question and on a good many other factors; and it is primarily this time interval which makes the task of deriving a statistical supply curve so much more difficult than that of a statistical demand curve. In deriving the demand curve for a given commodity, the question that we put to ourselves is, "What is the relation between the price for any given year (or period) and the consumption for the *same year?*" The statistical procedure suggested by this question is to correlate consumption with price for identical years or periods. If, after allowing for the "disturbing factors" affecting our variables, we obtain a negative correlation, suggesting that as prices rise consumption falls off, we are probably dealing with a true demand curve. But the question that we try to answer when we attempt to derive a supply curve, namely, "What is the relation between

[1] See chapters ii and iii.

changes in prices and changes in production to which they give rise?" does not suggest a safe statistical procedure, for the reason that we can not tell a priori which are the prices that have called forth any given production, whether they are present prices, past prices, or estimated future prices.

In reaching a decision on these questions some help may be derived from economic theory if we recall[2] that there is little or no theoretical justification for the assumption of the existence of negatively inclined supply curves in a régime of free competition. This means that we shall not go wrong often if we reject all negative correlations between prices and production on the ground that they can not generally lead to a true supply curve, and direct our attention primarily to the positive correlations between prices and production. But even this criterion will not always lead to the true supply curve, for the reason that a positive correlation between production and prices may often be obtained in several ways—say, by "lagging" prices behind production as well as by "lagging" production behind prices—and it is often difficult to decide on a priori grounds which is the proper method to follow. Furthermore, some commodities may be produced for several years at a stretch at approximately the same rate even though their prices have fallen to a mere fraction of their former level and the production is carried on at a loss, for the simple reason that to curtail production would mean a still greater loss. For example, one planting of sugar cane yields from 4 to 10 crops of sugar in Cuba, Hawaii, and Porto Rico. The production of sugar in such countries may, consequently, show little response to price changes from one year to the next.

The conclusion is, therefore, inescapable that, a positive correlation between the price and the production of a given commodity does not necessarily indicate a true supply relationship, just as a negative correlation between the price and the consumption of a given commodity does not necessarily indicate a true demand relationship. The statistical results must be con-

[2] See *supra*, pp. 113–14, 120.

firmed by "outside evidence"; that is, they must admit of a reasonable explanation in terms of the technology of the industry or the commodity under consideration.

If the "critical supply-price period," i.e., the period during which prices have the greatest effect on production, is known, the next question that has to be decided is whether to express the supply price of our commodity as a function of only one variable, the quantity produced, or as an explicit function not only of the quantity produced but also of the prices of the factors of production—the cost of materials, the interest paid on borrowed capital, the wages paid to workmen, etc. If the commodity under consideration be an agricultural commodity, the adoption of the second method would almost certainly mean that the supply price would be expressed as an explicit function of (1) the acreage planted, (2) the weather conditions, (3) the yield per acre, and (4) the other factors. The first method gives the dynamic law of supply in its simple form. The second method results in an expression for the dynamic law of supply comparable to "the dynamic law of demand in its complex form," or equations (3*b*) and (3*c*).

There can be no doubt but that the second method of approach is, in general, to be preferred, for it makes possible the use of the powerful tools of multiple and partial correlation to measure or to eliminate the effect of such disturbing factors as may be desired. However, in the problem immediately before us—how to deduce the supply curve for sugar—it will not be practicable to use the second method at the outset, because the "critical price period" and most of the necessary data are not known, or have to be determined, and for this preliminary work the first method, which involves the use of link relatives and trend ratios, is more practicable. Furthermore, it will be found, as the work progresses, that the simpler tools will yield results of approximately the same degree of accuracy as those obtained in the study of demand.

If the "critical supply-price period" is known, and if it co-

incides with the "critical demand-price period," this may mean that the same data must be used for deriving both the demand curve and the supply curve; and when this condition obtains there may result an approximation to the interesting relation between the two curves, to which reference has already been made.[3]

Let us assume that there is perfect negative correlation ($r = -1.0$) between the relative prices and the relative[4] consumption of a commodity for identical "points" in time. From such two perfectly correlated series we can get a perfect demand curve, that is, a negatively inclined curve which describes all the observations, without exception. Since, by assumption, the demand price is also the supply price, this means that the *same* price to which consumers are adjusting their consumption at any given time also "calls forth" a given production at some later time—say, one year hence. That is, the price for any (small) interval of time during any given year is positively correlated with the production for the corresponding interval in *the following year*. If disturbing factors be abstracted, the correlation will be not only positive, but also perfect, or $r = +1.0$. From such a pair of perfectly correlated series we can get a perfect supply curve, that is, a positively inclined supply curve which describes all the observations. It can be shown, however, that if the two series are to give perfect correlations both for demand and (by lagging one of them) for supply, they must be periodic functions of time. Two series may be perfectly correlated for demand; but unless they are each composed of perfect and uniform "cycles," they cannot be made to yield a perfect correlation for supply by the lagging process. Hence the assumptions that the "critical demand-price period" coincides with the "critical supply-price period," and that the correlations for both demand and supply are

[3] See *supra*, pp. 101–3.

[4] By the term *relative* prices or *relative* consumption is meant prices or consumption referred to some "normal" as a base, in order to allow for disturbing factors.

perfect, is equivalent to the assumption that the price series and the quantity series are each periodic functions of time (see pp. 101–3.

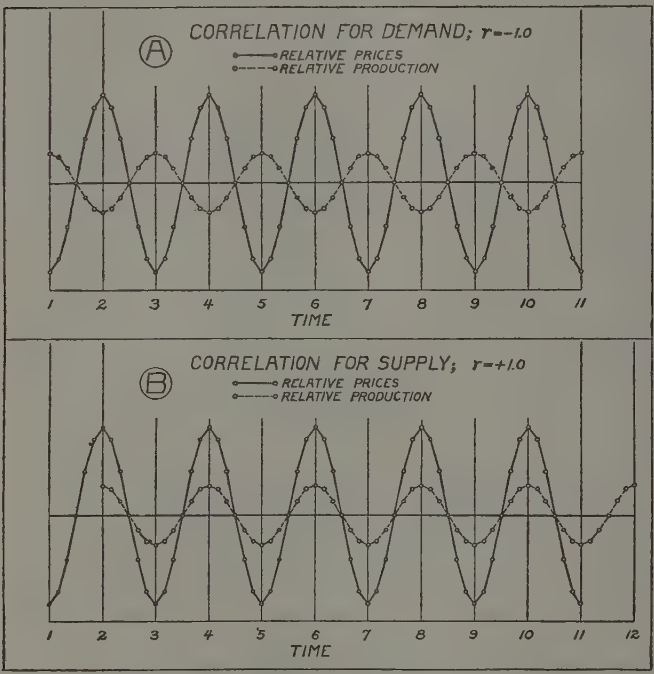

FIG. 28.—The "lag" method: correlations for demand and supply in a hypothetical self-contained economy, where the price and the quantity series are periodic functions of time.

This may be best illustrated by a hypothetical example. Figure 28*A* shows a perfect inverse correlation ($r = -1.0$) between the relative prices of a commodity for a series of years and the relative production for the same years. (Both prices and production are hypothetical.) From such perfectly correlated series we get a perfect demand curve. To every change in price

there is a corresponding change in consumption, without exception. The two series are simple periodic functions of time.

Figure 28*B* shows the same data as in Figure 28*A*, but with the production (consumption) series "lagging" behind the price series by one year. The correlation is perfect and positive ($r=+1.0$). The two series give a perfect supply curve. To every change in price there corresponds a given change in production, allowance being made for the *time interval* that must elapse before production can respond to price changes.

What makes possible the *perfect* correlation for supply is not the perfect correlation for demand, but the fact that the two series are each made up of uniform cycles. If this were not so, a perfect correlation for demand would not, by the lag method, also give a perfect correlation for supply.

We are now in a position to see why, under the conditions assumed, the elasticity of supply tends to be numerically equal to the elasticity of demand.[5] Recalling that in the previous example the correlation of the data for the law of demand is $r=-1.0$ and that the correlation of the data for the law of supply is $r=+1.0$, we have for the equations to the demand and supply curves:

$$\left.\begin{array}{l} Y=-\dfrac{\sigma_Y}{\sigma_X}X+B \\ \text{and} \\ Y=+\dfrac{\sigma_Y}{\sigma_X}X+C \end{array}\right\}, \qquad (29)$$

respectively, where σ_Y is the standard deviation of the prices, σ_X the standard deviation of production, and B and C are constants. Since

$$\eta=-\frac{\sigma_Y}{\sigma_X}\frac{Y}{X}$$

by (1), and

$$e=+\frac{\sigma_Y}{\sigma_X}\frac{Y}{X}$$

[5] See *supra*, pp. 101–3.

by (17), we obtain

$$|e| = |\eta| . \tag{30}$$

Of course, where the correlation between X (production) and Y (price) is not perfect for both demand and supply, the elasticity of supply will not be exactly equal to the elasticity of demand. *But if the method of fitting demand and supply curves be adopted which takes into consideration errors in both variables, e will tend to be approximately equal (numerically) to η even when the correlation between the variables is not perfect.*

This "lag" method of deriving the supply curve is due to Professor Moore, who has used it to derive the law of supply for potatoes[6]—a commodity with respect to which the United States is a self-contained economy. Using the method of trend ratios, Professor Moore finds that for the period 1900–1913 the correlation for the law of demand is $r = -0.95$ and the correlation for the law of supply is $r = +0.80$. Professor Moore does not give a detailed explanation of his method, but an examination of the way in which he uses the same data to obtain both the demand curve and the supply curve leaves no doubt as to the rationale of it.[7]

In connection with the application of the "lag" method to concrete problems, several points must be emphasized.

1. The assumptions on which it is based render invalid its application to problems in which these assumptions are not realized—at least approximately. When these are not realized to a fair degree of approximation, the correlation of price with production "lagged" by one year (or other period) does not always lead to the law of supply. Very often a higher positive correlation is obtained when the *prices* are "lagged" one year behind production, that is, when we correlate the production for the current year with the price for the *following* year.

[6] "A Moving Equilibrium of Demand and Supply," *Quarterly Journal of Economics*, XXXIX (May, 1925), 367–71.

[7] For an application of this method to the derivation of the world-demand and supply curves for sugar, see *infra*, pp. 164–68.

When we correlate the price for the current year with the production for the following year (or production for the current year with the price for the preceding year), we are assuming, in fact, that the high or low production for any year is the result of the producers' response to the high or low prices during the preceding year (or critical production period). When changes in weather conditions can be abstracted, and when the commodity under consideration is a semiperishable commodity like potatoes, such an assumption is not unreasonable, for neither the commodity itself nor its factors of production are under monopoly control, and there is a definite interval between one year's crop and the next. Thus, Professor Moore finds, from the statistics for 1900–1913, that for potatoes the correlation between the production-trend ratio of any given year and the price-trend ratio of the preceding year is $r = +0.80$.[8] Had he correlated the production trend ratio of the given year with the price trend ratio of the following year, that is, had he "lagged" prices behind production by one year, he would have found that the correlation is only $r = +0.38$. The assumption that producers of potatoes in any one year are influenced by the prices of the preceding year is, therefore, well borne out.

But such an assumption cannot safely be made when the commodity in question is a non-perishable world-commodity whose production is more or less continuous, or whose factors of production are in part or in whole under monopolistic control. For such a commodity it may well be that price follows production or that changes in price are synchronous with changes in production.

2. The "lag" method for deducing supply curves cannot, as a rule, be applied directly to unadjusted data of prices and production which are affected by many disturbing factors. The data must first be adjusted for these disturbing factors.

In short, the "lag" oracle is not infallible, when the necessary conditions (assumptions) are not realized; it will some-

[8] Moore, *op. cit.*, p. 368.

times be dumb, sometimes ambiguous, and sometimes lie. Nevertheless, it is not without its uses, especially in connection with a general study of the industry or commodity under consideration.

II. THE RELATION OF DATA TO METHOD

The three questions that present themselves as soon as we attempt to deduce the statistical law of supply for sugar are: (1) What shall be taken as the "supply" of sugar in the United States? (2) What is the critical supply-price period? (3) Which of the several price series is most logically related to the supply?

1. The sugar consumed in the United States is derived partly from domestic production (beet and cane), partly from imports from insular possessions on which no duty is levied, and partly from imports from the rest of the world on which the full duty is levied. Table I of Appendix III, shows the production, imports, and consumption of sugar in the United States for the years beginning July 1, 1902–14. The same data, together with the world-production, are also shown graphically in Figure 29. Which of these series—domestic production, imports from insular possessions, net imports from foreign countries, or the sum of the three (i.e., domestic consumption, approximately)—shall be taken as "the quantity supplied" for the purpose of deducing the statistical law of supply?

2. The critical supply-price period, if there be one, is affected by many factors. Some of these are related to the custom which prevails in the sugar industry of paying beet and cane farmers a basic price, plus a sliding scale increment, based on the price received by the factory for refined sugar.

In the beet-sugar industry, contracts are made between the factories and the growers as early as December, preceding planting season. The acreage planted may, therefore, depend to some extent upon the prices prevailing during the preceding winter.

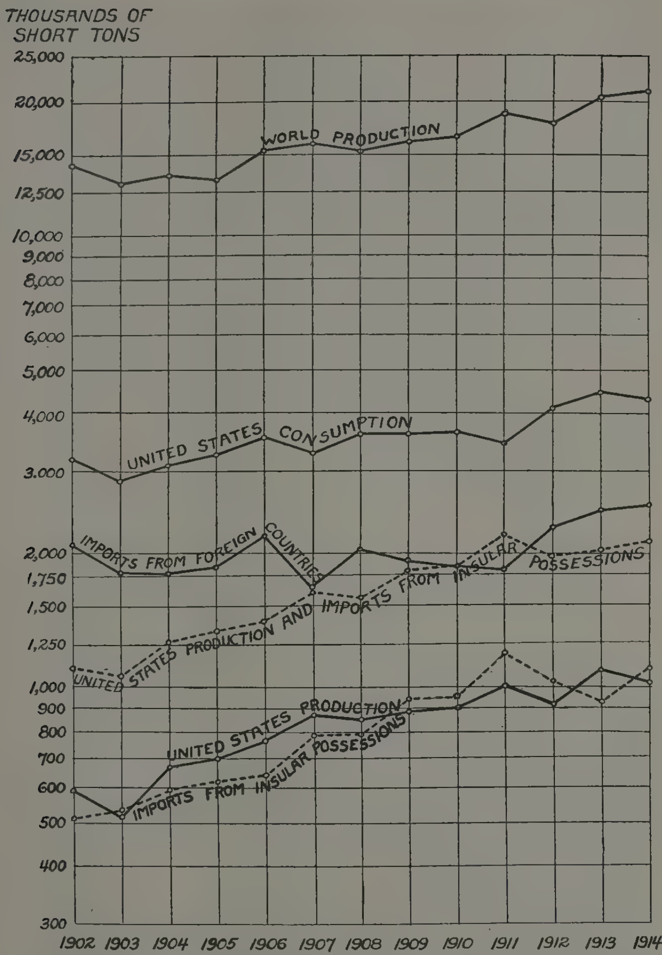

FIG. 29.—Sugar production, trade and consumption of the continental United States, and world-production, years beginning July 1, 1902–14.

The planting of beets takes place during April and May; and if the price of sugar is not high enough at this time, the contracts may be repudiated. Furthermore, if the price of sugar falls to an unprofitable level during the growing season, part of the acreage planted to beets may be abandoned.

In Louisiana, conditions are entirely different; there a single planting of cane yields two or three crops of sugar. The acreage devoted to cane cannot therefore be readily curtailed in the event of adverse price movements.

Both the cane crop of Louisiana and the beet crop of the entire country must be put through the mills each year. There can be no carry-over of cane or beets such as exists in Cuba and in other places. In both Louisiana and the rest of the country, but especially in the former, variations in production in some years may be due more to changes in weather conditions than to changes in the acreage planted.

In Porto Rico, Hawaii, Cuba, and the Philippines, from four to ten crops of sugar are harvested from each planting of cane. Production in these regions may therefore be even less responsive to changes in price than is the United States' production. Furthermore, if the price of sugar is considered unprofitable during the grinding season, the cane may be left in the fields until the following year. Planting is done both in the spring and fall so that high prices in either season may result in an increased acreage. The effect of the increase, however, may not be felt until one, two, or three years after the planting. Thus in the spring of 1923, the high price of sugar stimulated planting in the cane countries, and the result was heavy sugar production in 1925 and 1926.

Since the conditions obtaining in the several areas of production are far from uniform, what is the critical supply-price period? How shall the individual (monthly or weekly) price quotations be averaged so as to obtain the average price which has the greatest effect on production? Shall the average be for

the planting-season, the harvesting-season, the calendar year, the fiscal year?[9]

3. The various price series at our disposal also raise several questions. There are retail prices and wholesale prices; and of the latter, prices of raw sugar and prices of refined sugar. There are prices for the New York market and prices for other markets. Table II of Appendix III shows a basic price series—the New York wholesale prices of granulated sugar, by months, 1902–14. The same prices are also graphed in Figure 30. In Table II A of the same Appendix, averages of these monthly prices are shown for the years beginning July 1, for the eight months from October to May, and for the six months from January to June. Which of these prices are best for our purposes?

Some of these questions would not have arisen at all had the United States been a self-contained economy with respect to sugar, for then production would have been equal to consumption and there would not have existed such a complex of price relationships as are actually found in the sugar market. The demand-supply problem is much simpler in a closed economy.[10] In view of this it may be worth while to find out whether, in spite of the fact that the United States is not a closed economy with respect to sugar, the total consumption of sugar in any one year may not be looked upon as the "supply" called forth by the price for the preceding year (or period), and thus permit the use of the method applicable to a true closed economy?[11] For, should

[9] There is also the related question of the kind of average that is appropriate—the arithmetic mean, the geometric mean, the harmonic mean, the median, etc. For the discussion of this question see Holbrook Working, "Factors Determining the Price of Potatoes in St. Paul and Minneapolis," *University of Minnesota Agricultural Experiment Station, Technical Bulletin No. 10* (October, 1922), pp. 9, 10. In the present study, however, the question of the type of average is of minor importance when compared with those discussed above. We shall, therefore, use the unweighted arithmetic mean of the monthly price quotations.

[10] See pp. 101–3, 132–37.

[11] Of course, stocks carried over from one year to the next, as well as variations in the acreage planted and in weather conditions ought to be taken into account, but unfortunately, statistics on these subjects are not available for the entire period covered by this study of supply.

the "lag" method give good results in this experiment, may it not also throw some light on the various questions that have just been raised? Let us see.

In deducing the law of demand for sugar, four pairs of series have been used relating to consumption and prices for the calendar years 1890–1914. These were (1) the link relatives of the unadjusted data of consumption and prices, (2) the trend ratios of the unadjusted data of consumption and prices, (3) the link relatives of the adjusted data of consumption and prices, (4) the trend ratios of the adjusted data of consumption and prices.

To obtain a law of supply from each of the four pairs of series, all we have to do is to correlate the prices (link relatives or trend ratios) for the given years with the consumption (link relatives or trend ratios) for the following years, i.e., to "lag" consumption behind prices. The correlations are as follows:

(1) For the link relatives of the unadjusted data, $r=+0.43$, as compared with the correlation for demand of $r=-0.68$.
(2) For the trend ratios of the unadjusted data, $r=+0.01$, as compared with the correlation for demand of $r=-0.78$.
(3) For the link relatives of the adjusted data $r=+0.32$, as compared with the correlation for demand of $r=-0.67$.
(4) For the trend ratios of the adjusted data, $r=-0.10$, as compared with the correlation for demand of $r=-0.79$.

Reference to their standard errors shows that the correlations for supply are not statistically significant, with the probable exception of the first. It is clear, then, that the quantity of sugar consumed in any one year cannot be taken as equivalent to the quantity of sugar supplied in the same year. We must, therefore, break up the problem into its constituent parts. We must, in short, answer the various questions about the data that have been raised above. And we must do it under a serious handicap —lack of information on carry-over, acreage planted, and weather conditions.

Let it be admitted at the outset that under these conditions no purely a priori answers are possible. We must experiment

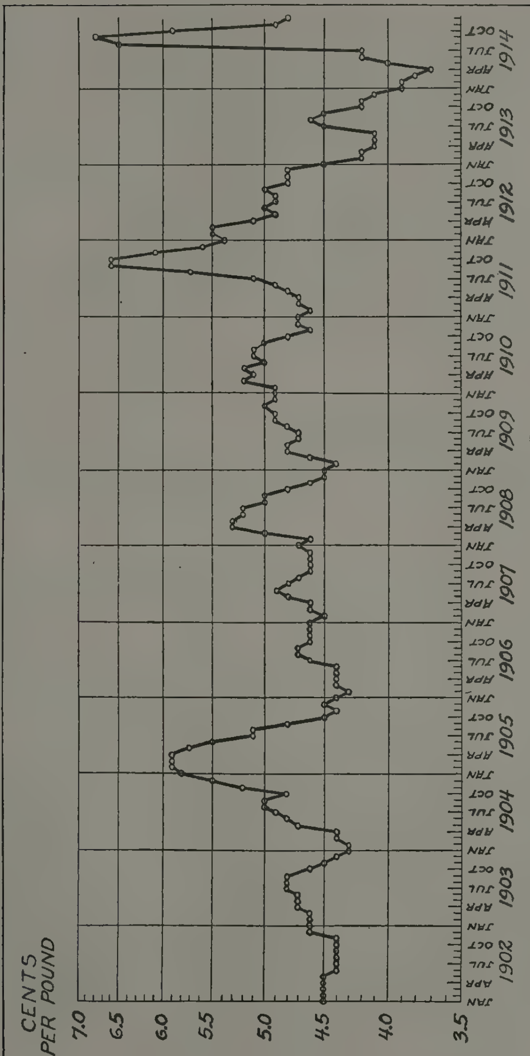

FIG. 30.—Average New York wholesale prices of granulated sugar, by months, 1902–14

with the various series relating to quantities and the various series relating to prices in order to determine which pair of price and quantity series gives the most significantly correlated results.

In the following pages we shall present the results of some of the many experiments that have been suggested by the problem of the law of supply of sugar. More specifically, we shall find the correlation of each of the three series:

X_1=domestic production (link relatives),
X_2=imports from insular possessions (link relatives), and
X_3=domestic production and imports from insular possessions (link relatives)

with

Y_1=average of the prices for the twelve months of the fiscal year July to June (link relatives),
Y_2=average of the prices for the eight months October to May (link relatives), and
Y_3=average of the prices for the six months January to June (link relatives)

for various lags. The three quantity series (together with other related series) are graphed in Figure 29, and the three price series are graphed in Figure 31.

From these correlations we shall derive several supply curves and show their relation to the world supply curve for sugar.

The selection of the three periods for which to average the monthly prices has been dictated by the following considerations: (1) The fiscal year beginning July 1 is the year for which the production statistics are published by the Department of Agriculture. (2) The eight months October to May are the months during which sugar is being harvested in one or more of the great producing-regions of the world. (3) The six months from January to June are the months during which practically all of our imports of sugar reach this country. Furthermore, the average price for these six months is not markedly different from

that for the first half of this period, or for the three months from January to March, so that any conclusion which applies to the former also applies to the latter. This is important in view of the fact that the three months in question immediately pre-

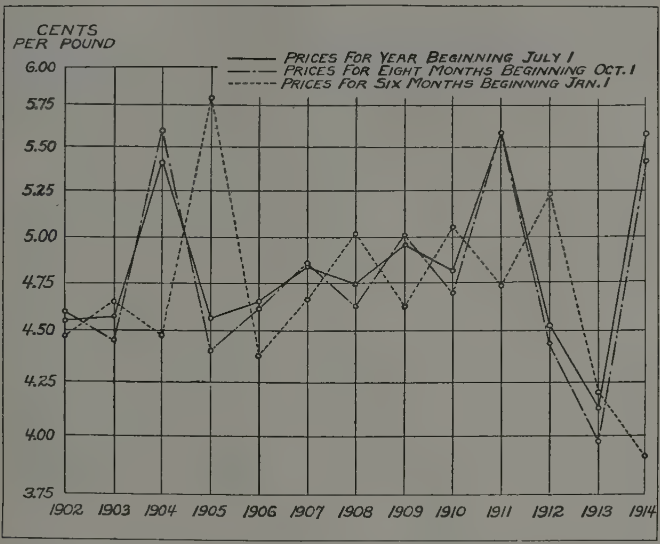

FIG. 31.—Averages of the New York monthly prices of granulated sugar for the year beginning July 1 (Y_1), for the eight months beginning October 1 (Y_2), and for the six months beginning January 1 (Y_3), 1902–14.

cede the planting-season and they may, therefore, constitute a "critical supply-price period" for United States production.

Only the method of link relatives will be used in the following correlations, as the many series which had to be studied—and of these only a small proportion are treated in the following pages—made it impracticable to use any but the simplest methods.

Furthermore, as it will be desirable in a later chapter to ap-

ply some of our results to a tariff problem, the analysis will have to be confined to a period during which there was no change in the tariff on sugar. The longest period since 1890 fulfilling this condition is that from December 27, 1903, to March 1, 1914, during which the full duty on 96° centrifugals remained constant at 1.685 cents per pound. The beginning of this period is the date on which the act of December 17, 1903 became effective, giving Cuban sugar a 20 per cent reduction from the full duty rate; the end of this period is the date on which the act of October 3, 1913, became effective, reducing the duty on all sugar imports by approximately 25 per cent. However, in most of the correlations which follow, the analysis will cover a slightly longer period—that from July 1, 1903, to June 30, 1914—which is taken as equivalent to the foregoing. This is done for the following reasons:

1. All of the quantity series and some of the price series used in this study of supply are given by fiscal years beginning July 1. To have excluded from the computations the data for the entire fiscal years 1903 and 1913 on the ground that during a part (less than half) of each of these years a different tariff was in force would have meant a reduction in the number of observations from 11 to 9 and a corresponding decrease in the significance of any correlation coefficients or other constants deduced from them.

2. To have excluded the data for the two years in question would have still further reduced the number of observations available for experimentation with the "lag" method.

3. A graphic comparison of all the data shows that the observations for these two years fluctuate in about the same way as all other observations (see, for example, Figs. 32, 33, and 34).

In Tables I, I A, II, and II A of Appendix III, the data are presented for a somewhat longer period than the eleven years beginning July 1, 1903, in order to allow for the loss of end years incident to the use of the link relative and the "lag" methods. The exact period covered by each correlation is given in Tables VI–IX of the text.

III. THE LAW OF SUPPLY FOR SUGAR DERIVED FROM DOMESTIC PRODUCTION X_1

1. CORRELATIONS BETWEEN DOMESTIC PRODUCTION X_1 AND THE THREE PRICE SERIES, Y_1, Y_2, AND Y_3

Let us consider first the correlations between fiscal-year prices Y_1 and domestic production X_1.

The correlation between Y_1 for any given year and X_1 for the following year is $r = -0.51$. This is a surprising result, for, from what is known of the methods by which beet and cane farmers are paid,[12] we should expect high prices to call forth high production, and low prices low production; that is, we should expect a *positive* correlation between prices for any given year and the production for the following year. The negative sign of the correlation suggests that we are on the wrong track, for the short-time supply curve, the one which we are attempting to derive, cannot normally have a negative slope,[13] even if the long-time supply conditions are such that an increase in output is accompanied by a decrease in the price. We must, therefore, try other ways of correlating the series under consideration.

The correlation between Y_1 for any given year and X_1 for the same year is positive, r being $+0.49$; and the correlation between Y_1 for any given year and X_1 for the *preceding* year is negative, r being -0.59. These results indicate that the domestic supply curve must be deduced from prices which fall within the same year as production. This does not mean, however, that

[12] "*The Price of Sugar Affects the Cost of Producing Sugar.* The reason why so great a disparity hinges on the choice of year calls for explanation. It is due chiefly to the custom of paying for raw material, cane or beets, a price dependent on the price of sugar. In Cuba the centrals pay the colonos for cane in terms of so many pounds of sugar (at the f.o.b. Cuban port price) per ton of cane, the price being the average of the prices within two weeks of the delivery of the cane. A similar method prevails in Louisiana and Porto Rico. Beet farmers are paid a basic price plus a sliding scale increment based on the price received by the factory for refined sugar. In the Hawaiian Islands also laborers receive a basic wage plus a bonus graduated according to the price of sugar" (Philip G. Wright, *Sugar in Relation to the Tariff*, pp. 139–40).

[13] See pp. 113–14, and 120.

the average of the prices for the twelve months of the fiscal year will necessarily give the best results. The supply-demand relationship may be such that the price for a particular month or season may have no measurable influence on production, or that the price for some months may have an important influence on

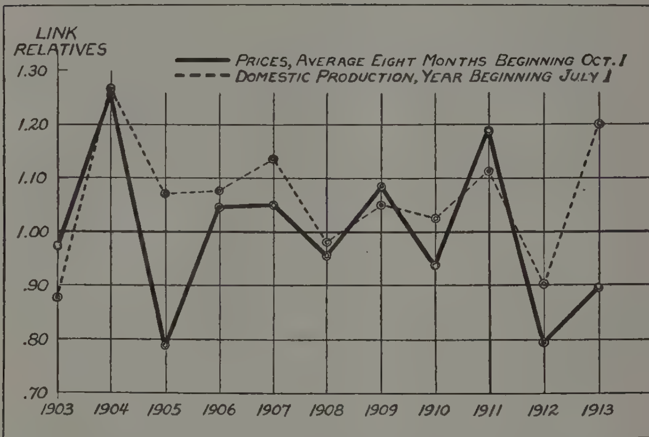

FIG. 32.—Positive correlation between the link relatives of the average wholesale prices of sugar for the eight months beginning October 1 (Y_2) and the link relatives of the annual United States production of sugar (X_1), 1903–13.

cane production without appreciably affecting beet production, and conversely.

Table VI shows the results obtained by substituting, successively, for the fiscal-year prices (Y_1) used in the foregoing correlations the prices for the eight months October to May (Y_2), and the prices for the six months January to June (Y_3). The reason for using these series has already been stated.[14] One of these correlations—that between Y_2 (1903) and X_1 (1903)—is also shown graphically in Figure 32.

[14] See pp. 144–45.

In interpreting the data of Table VI, it should be borne in mind that the fact that a given average price is not based on all of the twelve monthly quotations does not necessarily mean that such an average is less typical of the (unknown) true average supply-price than is the one based on the quotations for all the twelve months. In fact, it may be the most logical average to use for our purposes if the quotations on which it is based are for the months during which the price-making forces are most active.

TABLE VI

CORRELATION COEFFICIENTS BETWEEN DOMESTIC PRODUCTION X_1 AND THE THREE PRICE SERIES Y_1, Y_2, AND Y_3, FOR VARIOUS LAGS*

Series Correlated	Coefficient of Correlation, r
Y_1 (1903) and X_1 (1904)	−0.51
Y_1 (1903) and X_1 (1903)	+0.49
Y_1 (1904) and X_1 (1903)	−0.59
Y_2 (1903) and X_1 (1904)	−0.51
Y_2 (1903) and X_1 (1903)	+0.54
Y_2 (1904) and X_1 (1903)	−0.63
Y_3 (1904) and X_1 (1904)	−0.40
Y_3 (1904) and X_1 (1903)	+0.60
Y_3 (1904) and X_1 (1902)	−0.90

*The figures in the brackets indicate the *first* year of each of the correlated series under consideration. Thus, Y_1 (1903) means the average price for the fiscal year beginning July 1, 1903; X_1 (1904), the domestic production for the fiscal year beginning July 1, 1904; Y_2 (1903), the average price for the eight months beginning October 1, 1903 (i.e., October 1, 1903, to May 30, 1904); Y_3 (1904), the average price for the six months beginning January 1, 1904. Both prices and production are in terms of link relatives. Each pair of correlated series covers a period of ten or eleven years, depending on whether a lag of one year was, or was not, used in the correlation. See Tables I A and II A of Appendix III.

Table VI and the data on which it is based suggest the following conclusions:

(1) Whichever prices are used, a positive correlation between sugar prices and sugar production (domestic) is obtained only when the two series are synchronous. When either series is "lagged" behind the other by one year the correlation is negative. The "lag" method, which is likely to give good results in a

self-contained economy, is thus found to be inapplicable to the problem under consideration.

(2) A better understanding of the relation between changes in price and changes in production might have been obtained from a study of the relation between changes in price and changes in the *acreage planted to sugar;* but, as was already pointed out, no data on this subject can be found previous to 1914–15.

(3) There appears to be no significant correlation between changes in the acreage harvested (beets) and changes in the average price for the quarter immediately preceding the planting season, or between changes in the acreage harvested and any of the price series Y_1, Y_2, or Y_3.

(4) If the relation between the domestic production and any of the three price series be a causal one, then high or low prices are not the sole cause of high or low production, for r has a value of only —0.5 or —0.6. Factors other than price may often exert a deciding influence on domestic production, and these factors ought to be accounted for in any investigation having in view the forecasting of production or prices.

(5) If there be a causal relation between high or low prices for any given year (either Y_1, Y_2, or Y_3) and high or low production *in the same year,* it is evident that changes in prices may effect changes in production *after the crop has been planted.* How is this possible? The hypothesis that suggests itself is this: When prices are high, beet farmers will spend more time and labor in cultivating their beets, thus increasing the yield, and cane farmers will cut more cane; and conversely when prices are low. Cane farmers, especially, ought to be able to adjust their output to prevailing prices, either by varying the amount of cane left for seed or by advancing or retarding the harvesting-season. For example, if the price of sugar is high, cane in the tropics may be harvested when a year old; and if the price of sugar is low, the harvesting may be delayed for several months. But any

adjustment of production to prevailing prices can, of course, take place within certain limits only.[15]

(6) Judged by the magnitude of the three positive correlation coefficients, Y_2 and Y_3 appear to have a somewhat greater effect on X_1 than does Y_1. But the difference between any two of these coefficients is not statistically significant.[16]

2. EQUATIONS BETWEEN DOMESTIC PRODUCTION X_1 AND THE THREE PRICE SERIES, Y_1, Y_2, AND Y_3

The "lines of best fit," or the "lines of mutual regression," giving the most probable relation between changes in domestic production (X_1) and changes in each of the three price series are, for the positive correlations:

$$Y_1 = 0.9899X_1 - 0.055 \tag{31}$$

$$Y_2 = 1.5142X_1 - 0.613 \tag{32}$$

$$Y_3 = 1.5055X_1 - 0.606 \tag{33}$$

the origins being 0,0.

These equations mean that, based on the experience from 1903 to 1913, an increase (or decrease) of one *point* (or, roughly, 1 per cent) in the link relative of the domestic production (beet and cane) is, *on the average,* associated with an increase (or decrease) of 0.99 of one *point* in the relative price for the fiscal year and with 1.51 *points* in the relative prices for October to May, and January to June, respectively. Another way of saying

[15] This hypothesis is considered reasonable by Dr. C. O. Townsend, chief of the Sugar Division, U.S. Tariff Commission, and Mr. Lester D. Johnson, U.S. Sugar Association, Washington, D.C. I am indebted to these gentlemen for advice on the selection of some of the series employed and for information on the technical phases of the industry. I am also indebted to Professor John H. Cover, University of Denver, for valuable information.

[16] When the observations are as few as those on which these correlations are based, it is not advisable to give "probable errors," though recourse may be had to the powerful tools provided in R. A. Fisher's *Statistical Methods for Research Workers* (London, 1925), for testing the significance of constants computed from small samples. Such conclusions as are attached in the following pages from the various correlations for supply are based more on the variety of methods and data used than on the significance of any coefficient of correlation (or other constant) with respect to its probable error.

the same thing is that an increase (or decrease) of one *point* in the relative price for the fiscal year is, *on the average*, associated with an increase (or decrease) of 1/0.9899=1.01 *points* in the relative production; and that an increase (or decrease) of one *point* in either the relative price for October to May or the relative price for January to June is, *on the average*, associated with an increase (or decrease) of only 1/1.51=0.66 of a *point* in the relative production.

By means of these equations it is possible to estimate the probable change in domestic production corresponding to a given change in any of the three price series, and conversely.[17]

The same equations may also be readily expressed in terms of absolute quantities and prices. They then afford a measure of the shifting of the domestic supply curve as a result of dynamic changes.[18]

3. COEFFICIENT OF ELASTICITY OF THE DOMESTIC SUPPLY

The coefficient of the elasticity of supply is

$$e=\frac{dX}{dY}\frac{Y}{X} \tag{34}$$

where X and Y are the link relatives of production and prices, respectively.[19]

Applying this definition to equations (31), (32), and (33), we have

$$e_{x_1y_1}=\frac{1}{0.9899}\frac{Y_1}{X_1}, \tag{35}$$

$$e_{x_1y_2}=\frac{1}{1.5142}\frac{Y_2}{X_1}, \tag{36}$$

$$e_{x_1y_3}=\frac{1}{1.5055}\frac{Y_3}{X_1}, \tag{37}$$

[17] For an illustration of the method as applied to the law of demand see pp. 39–40, 74.

[18] For an illustration of the method as applied to the law of demand, see pp. 40–42, 74–76.

[19] The proof is the same as for η which is given on p. 45.

respectively. These are all increasing functions of price; the elasticity of supply is higher for high prices than for low prices.[20]

At the point on each supply curve whose co-ordinates are the arithmetic means of the link relatives,[21] the coefficients have the following values:

$$e_{X_1Y_1}=+0.95\ , \tag{35a}$$

$$e_{X_1Y_2}=+0.62\ , \tag{36a}$$

$$e_{X_1Y_3}=+0.63\ . \tag{37a}$$

These figures mean that, based on the average experience of 1903–13, an increase (or decrease) of 1 per cent in the average price of sugar for any year is accompanied by an increase (or decrease) of 0.95 of 1 per cent in the domestic production for

[20] The elasticity of supply is, by definition,

$$e=\frac{dX}{dY}\cdot\frac{Y}{X}=\frac{1}{b}\,\frac{a+bX}{X}=\frac{a}{bX}+1\ ,$$

where $Y=a+bX$ is the supply function.

In this function $b>0$ always, while $a\lesseqgtr 0$. Whether e is an increasing, constant, or decreasing function of X depends, therefore, upon whether $a\lesseqgtr 0$. Since, in equations (31), (32), and (33), $a<0$, the coefficients of the elasticity of supply deduced from them must be increasing functions of the output (X). And since the output is an increasing function of the price (Y) the elasticities of supply are also increasing functions of Y.

If the straight line $Y=a+bX$ is used to represent the *demand* function, $a>0$, $b<0$. The elasticity of demand, η, then becomes

$$\eta=-\frac{a}{|b|X}+1\ .$$

Since the price of a commodity can never fall to zero, $|a|>|bX|$. Like the elasticity of supply, the elasticity of demand is an increasing function of X. But since the elasticity of demand is negative for all values of X, this means that its *absolute* value decreases as X increases. That is to say, when the law of demand is represented by a straight line, the elasticity of demand (absolute value) is higher for low consumption or high prices than for high consumption or low prices. However, the conclusions reached in this study regarding the changes in the elasticity of demand as we go from one point to another on the demand curve, were never based on straight-line demand functions only. See Figures 13 and 23.

[21] The arithmetic means of the four variables are: $\overline{X}_1=1.063$, $\overline{Y}_1=0.998$, $\overline{Y}_2=0.997$, and $\overline{Y}_3=0.995$.

the same year; that an increase (or decrease) of 1 per cent in the average price for the eight months October to May is accompanied by an increase (or decrease) of 0.62 of 1 per cent in production; and a change of 1 per cent in the average price for the six months January to June has the same effect[22] as a change of 1 per cent in the average price for October to May.

Considering, however, the paucity of observations, it is probable that there is no statistically significant difference between these coefficients.

IV. THE LAW OF SUPPLY FOR SUGAR DERIVED FROM IMPORTS FROM INSULAR POSSESSIONS X_2

I. CORRELATIONS BETWEEN IMPORTS FROM INSULAR POSSESSIONS X_2 AND THE THREE PRICE SERIES, Y_1, Y_2, AND Y_3

The correlations between each of the three price series and the imports from our insular possessions are shown in Table VII. One of these—that between Y_2 (1903) and X_2 (1903)—is also represented graphically in Figure 33.

From Table VII it will be observed that—

(1) The *signs* of the various coefficients of correlation are the same as the signs of the corresponding coefficients in Table VI; that is to say, whichever price series is used, a positive correlation between sugar prices and imports from insular possessions is obtained only when the two series are synchronous. When either series is "lagged" behind the other by one year, the correlation is negative. The "lag" method which gives good results under certain conditions is thus found to be inapplicable to imports from insular possessions as well as to domestic production.

(2) All the coefficients are higher than the corresponding coefficients in Table VI, which means that imports from insular possessions are more closely responsive to price changes than is domestic production. This, however, is probably due to our failure to treat separately the effect of various price changes on

[22] That is, on the production for the year *ending* in the same June.

domestic beet production and cane production and on the exports from each of our insular possessions—Hawaii, the Philippines, and Porto Rico.

TABLE VII

CORRELATION COEFFICIENTS BETWEEN IMPORTS FROM INSULAR POSSESSIONS X_2, AND THE THREE PRICE SERIES, Y_1, Y_2, AND Y_3, FOR VARIOUS LAGS*

Series Correlated	Coefficient of Correlation (r)
Y_1 (1903) and X_2 (1904)	−0.07
Y_1 (1903) and X_2 (1903)	+0.75
Y_1 (1904) and X_2 (1903)	−0.38
Y_2 (1903) and X_2 (1904)	−0.11
Y_2 (1903) and X_2 (1903)	+0.72
Y_2 (1904) and X_2 (1903)	−0.41
Y_3 (1904) and X_2 (1904)	−0.03
Y_3 (1904) and X_2 (1903)	+0.64
Y_3 (1904) and X_2 (1902)	−0.71

*The figures in the brackets indicate the *first* year of each of the correlated series under consideration. Thus, Y_1 (1903) means the average price for the fiscal year beginning July 1, 1903; X_2 (1904), the imports from the insular possessions for the fiscal year beginning July 1, 1904; Y_2 (1903), the average price for the eight months beginning October 1, 1903 (i.e., October 1, 1903 to May 30, 1904); Y_3 (1904), the average price for six months beginning January 1, 1904. Both prices and imports are in terms of link relatives. Each pair of correlated series covers a period of ten or eleven years, depending on whether a lag of one year was, or was not, used in the correlation. See Tables I A and II A of Appendix III.

(3) If there be a causal relation between high and low prices for any given year and imports from insular possessions in the same year, it probably means that when domestic prices are high, sugar-growers in our insular possessions will cut, grind, and export a greater part of their crop than when prices are low.[23]

2. EQUATIONS BETWEEN IMPORTS FROM INSULAR POSSESSIONS X_2 AND THE THREE PRICE SERIES, Y_1, Y_2, AND Y_3

The "line of best fit" giving the most probable relation between changes in imports from insular possessions (X_2) associ-

[23] See pp. 140, 150–51.

ated (positively) with changes in each of the three price series are:

$$Y_1 = 0.8890X_2 + 0.052\ , \qquad (38)$$
$$Y_2 = 1.2217X_2 - 0.302\ , \qquad (39)$$
$$Y_3 = 1.2973X_2 - 0.380\ , \qquad (40)$$

the origins being at 0,0.

These equations mean that, based on the experience from 1903 to 1913, an increase (or decrease) of one *point* (or roughly

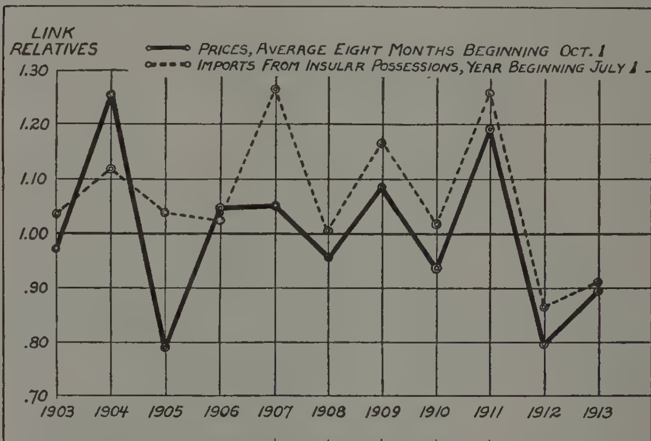

FIG. 33.—Positive correlation between the link relatives of the average wholesale prices of sugar for the eight months beginning October 1 (Y_2) and the link relatives of the annual imports of sugar into the United States from insular possessions (X_2), 1903–13.

1 per cent) in the link relative of the imports from insular possessions is, *on the average,* associated with an increase (or decrease) of 0.89 of 1 *point* in the relative price for the fiscal year; with 1.22 *points* in the relative price for October to May; and with 1.30 *points* in the relative price for January to June. Another way of saying the same thing is that an increase (or decrease) of one *point* in the relative price for the fiscal year is as-

sociated with an increase (or decrease) of $1/0.8890=1.12$ *points* in the relative imports; that an increase (or decrease) of one *point* in the relative price for October to May is, *on the average,* associated with an increase (or decrease) of only $1/1.2217=0.82$ of a *point* in the relative imports; and that an increase (or decrease) of one *point* in the relative price for January to June is, *on the average,* associated with an increase (or decrease) of only $1/1.2973=0.77$ of a *point* in the relative imports.

By means of these equations it is possible to estimate the probable change in the imports from our insular possessions corresponding to a given change in any of the three price series and conversely.[24]

The same equations may also be readily expressed in terms of absolute imports and prices. They then afford a measure of the shifting of the supply from our insular possessions as a result of dynamic changes.[25]

3. COEFFICIENT OF ELASTICITY OF THE SUPPLY FROM OUR INSULAR POSSESSIONS

Applying the definition of the elasticity of supply given in (34) to equations (38), (39), and (40), we have—

$$e_{x_2 y_1}=\frac{1}{0.8890}\frac{Y_1}{X_2}, \tag{41}$$

$$e_{x_2 y_2}=\frac{1}{1.2217}\frac{Y_2}{X_2}, \tag{42}$$

$$e_{x_2 y_3}=\frac{1}{1.2973}\frac{Y_3}{X_2}, \tag{43}$$

The first of these coefficients is a *decreasing function* of price; the next two are *increasing functions* of price. (See note 20, p. 153.)

[24] For an illustration of the method as applied to the law of demand, see pp. 39–40, 74.

[25] For an illustration of the method as applied to the law of demand, see pp. 40–42, 74–76.

At the point on the supply curve whose co-ordinates are the arithmetic means of the link relatives,[26] the coefficients have the following values:

$$e_{X_2Y_1} = +1.05 \,, \tag{41a}$$

$$e_{X_2Y_2} = +0.77 \,, \tag{42a}$$

$$e_{X_2Y_3} = +0.72 \,. \tag{43a}$$

These figures mean that based on the average experience of 1903–13, an increase (or decrease) of 1 per cent in the average price of sugar for any year is accompanied by an increase (or decrease) of 1.05 per cent in the imports from our insular possessions; an increase (or decrease) of 1 per cent in the average price for the eight months October to May is accompanied by an increase (or decrease) of 0.77 of 1 per cent in our imports from our insular possessions; while an increase (or decrease) of 1 per cent in the average price for the six months January to June is accompanied by an increase (or decrease) of only 0.72 of 1 per cent in the imports from our insular possessions.[27] Considering, however, the paucity of observations, it is probable that there is no significant difference between these coefficients.

V. THE LAW OF SUPPLY FOR SUGAR DERIVED FROM DOMESTIC PRODUCTION AND IMPORTS FROM INSULAR POSSESSIONS X_3

I. CORRELATIONS BETWEEN DOMESTIC PRODUCTION PLUS IMPORTS FROM INSULAR POSSESSIONS X_3 AND THE THREE PRICE SERIES, Y_1, Y_2, AND Y_3

The correlation between each of the three price series and the supply of sugar derived from domestic production and imports from insular possessions are shown in Table VIII. One of these—that between Y_2 and X_3—is also shown in Figure 34.

The conclusions suggested by Table VIII might have been foreseen from the consilience of the results of Table VI and Table VII. They are:

[26] The arithmetic mean of X_2 is $\overline{X}_2 = 1.064$. The arithmetic means of the other variables are given in footnote 21.

[27] That is, in the imports for the year ending in the same June.

TABLE VIII

CORRELATION COEFFICIENTS BETWEEN DOMESTIC PRODUCTION PLUS IMPORTS FROM INSULAR POSSESSIONS X_3 AND THE THREE PRICE SERIES, Y_1, Y_2, AND Y_3, FOR VARIOUS LAGS*

Series Correlated	Coefficient of Correlation (r)
Y_1 (1903) and X_3 (1904)	−0.30
Y_1 (1903) and X_3 (1903)	+0.76
Y_1 (1904) and X_3 (1903)	−0.54
Y_2 (1903) and X_3 (1904)	−0.33
Y_2 (1903) and X_3 (1903)	+0.77
Y_2 (1904) and X_3 (1903)	−0.58
Y_3 (1904) and X_3 (1904)	−0.23
Y_3 (1904) and X_3 (1903)	+0.75
Y_3 (1904) and X_3 (1902)	−0.94

* The figures in the brackets indicate the *first* year of each of the correlated series under consideration. Thus, Y_1 (1903) means the average price for the fiscal year beginning July 1, 1903; X_3 (1904), domestic production and imports from our insular possessions for the fiscal year beginning July 1, 1904; Y_2 (1903), the average price for eight months beginning October 1, 1903 (i.e., October 1, 1903 to May 30, 1904); Y_3 (1904), the average price for the six months beginning January 1, 1904. Both prices and domestic production and imports from insular possessions are in terms of link relatives. Each pair of correlated series covers a period of ten or eleven years, depending on whether a lag of one year was, or was not, used in the correlation. See Tables I A and II A of Appendix III.

(1) Domestic production and imports from insular possessions taken together are quite responsive to price changes. However, no significantly different coefficients are obtained by substituting Y_2 or Y_3 for Y_1.

(2) Production and imports from insular possessions for any one year respond to price changes in the same year. This does not mean, however, that the same prices have the same effect on the imports from the various sources. High prices for January to June, for example, may call forth big imports from Hawaii and the Philippines, but have little effect on the imports from Porto Rico.[28]

(3) The positive, and for our purposes the *significant*, co-

[28] The author is indebted to Mr. Lester D. Johnson, United States Sugar Association, Washington, D.C., for pointing out this fact.

efficients of correlation are somewhat higher than the corresponding coefficients in Tables VI and VII. This indicates that slightly better results are obtained by combining domestic production and imports from the insular possessions than by treating them separately.

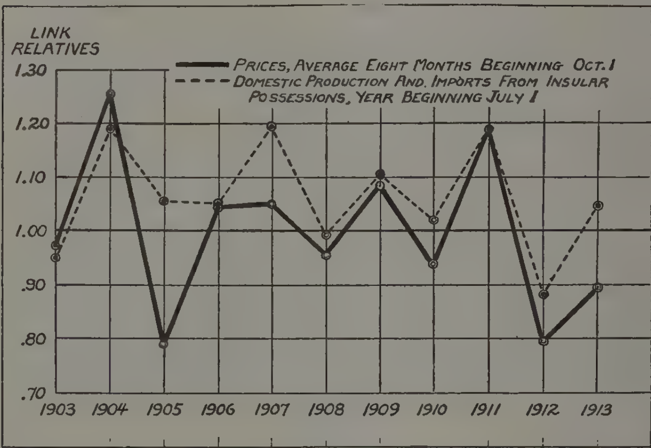

FIG. 34.—Positive correlation between the link relatives of the average wholesale prices of sugar for the eight months beginning October 1 (Y_2) and the link relatives of the combined annual United States sugar production and imports from insular possessions (X_3), 1903–13.

2. EQUATIONS BETWEEN DOMESTIC PRODUCTION PLUS IMPORTS FROM INSULAR POSSESSIONS X_3 AND THE THREE PRICE SERIES, Y_1, Y_2, AND Y_3

The "lines of best fit" giving the most probable variation between changes in the supply derived from domestic production and imports from insular possessions (X_3) and changes in each of the three price series are:

$$Y_1 = 1.2016X_3 - 0.284 , \tag{44}$$

$$Y_2 = 1.6110X_3 - 0.712 , \tag{45}$$

$$Y_3 = 1.6796X_3 - 0.787 , \tag{46}$$

the origins being at 0,0. These equations, which are based on the experience from 1903–13, mean that an increase (or decrease) of one *point* (or roughly 1 per cent) in X_3 is, *on the average,* associated with an increase (or decrease) of 1.20 *points* in Y_1, with 1.61 *points* in Y_2, and with 1.68 *points* in Y_3. Another way of saying the same thing is that an increase (or decrease) of one *point* in the relative price for the fiscal year is, *on the average,* associated with an increase (or decrease) of $1/1.2016=0.83$ of one *point* in the relative supply from the two sources in question. When the relative price for the eight months October to May is substituted for Y_1, the change is $1/1.6110=0.62$ of one *point.* When the relative price for January to June is substituted for Y_1, the change is only $1/1.6796=0.60$ of one point.

These equations enable us to estimate the probable change in the supply derived from domestic production and imports from insular possessions corresponding to a given change in any of the three price series considered, and conversely.

As has already been pointed out, the same equations may also be readily expressed in terms of absolute quantities and prices. They then afford a measure of the shifting of the supply curve deduced from X_3 as a result of dynamic changes.

3. COEFFICIENT OF ELASTICITY OF THE COMBINED SUPPLY DERIVED FROM DOMESTIC PRODUCTION AND IMPORTS FROM INSULAR POSSESSIONS

The coefficients of the elasticity of supply deduced from equations (44), (45), and (46) are:

$$e_{x_3y_1}=\frac{1}{1.2016}\,\frac{Y_1}{X_3}\,, \tag{47}$$

$$e_{x_3y_2}=\frac{1}{1.6110}\,\frac{Y_2}{X_3}\,, \tag{48}$$

$$e_{x_3y_3}=\frac{1}{1.6796}\,\frac{Y_3}{X_3}\,, \tag{49}$$

respectively. These are all increasing functions of price. (See note 20, p. 153.) The elasticity of supply from the two sources under consideration is higher for high prices than for low prices.

At the point on the supply curve whose co-ordinates are the arithmetic means of the link relatives,[29] the coefficients have the following values:

$$e_{x_3 y_1} = +0.78\,, \tag{47a}$$

$$e_{x_3 y_2} = +0.58\,, \tag{48a}$$

$$e_{x_3 y_3} = +0.56\,. \tag{49a}$$

These figures mean that, based on the average experience of 1903–13, an increase (or decrease) of 1 per cent in the *average* price of sugar for any year is accompanied by an increase (or decrease) of 0.78 of 1 per cent in the supply from domestic production and imports from insular possessions for the same year; that an increase (or decrease) of 1 per cent in the *average* price for the eight months October to May is accompanied by an increase (or decrease) of only 0.58 of 1 per cent in the supply from the two sources; and that an increase (or decrease) of 1 per cent in the *average* price for the six months of January to June is associated with an increase or decrease of only 0.56 of 1 per cent in the supply from the same sources.[30]

VI. THE LAWS OF WORLD SUPPLY AND DEMAND FOR SUGAR

Having deduced the law of supply for that part of our total consumption which comes from domestic and non-dutiable sources, the next logical step is to derive the law of supply for that part of our consumption which comes from all other sources and which, during 1903–13, accounted for between 64 and 48 per cent of our annual consumption. More specifically, this means that we ought next to correlate changes in the price of sugar in the United States with (1) changes in our imports from Cuba, which are subject to a 20 per cent tariff preferential, and

[29] The arithmetic mean of X_3 is $\overline{X}_3 = 1.061$. The arithmetic means of the other variables are given in footnote 21, p. 153.

[30] That is, in the production and imports for the year ending in the same June.

(2) changes in our imports from all full-duty-paying countries. But the statistics of Cuban exports and production do not show any appreciable correlation with the United States prices for the period 1903–13, at least when the method of link relatives is used. In fact, it appears that Javan production of sugar is much more highly correlated with the United States prices than are either the Cuban exports to the United States or the Cuban production.

This may be due either to the limitations of the statistical methods employed in this book, which cannot iron out the effects of such *local* disturbing forces as seem to have been affecting Cuban imports and exports during the period under consideration, or to our failure sufficiently to take into consideration the effects of changes in world-production.

We shall provisionally adopt the latter hypothesis for the reason that, the world being the only self-contained economy with respect to sugar, any attempt to study only a part of this economy—Cuba, for example—is to neglect the whole complex of price-making forces; is to fail to consider "the totality of occurrences which are in economic reality inexplicably connected with one another." The United States does not purchase the entire sugar crop of Cuba. Part of it must always compete in the world's markets with supplies from other sources. It is probable, therefore, that changes in world-production have some influence on conditions of Cuban supply. So long as we confined our study to the United States and its possessions, which are protected by a high tariff wall, we could abstract changes in world-production and still get broad, useful results for our purposes. But, to judge from the apparent lack of correlation between Cuban production or Cuban exports to the United States and New York prices, it is probably impossible to make this abstraction in the analysis of the Cuban statistics. We proceed, therefore, to an analysis of the *world* supply and demand for sugar.

I. CORRELATIONS BETWEEN WORLD-PRODUCTION X_w AND NEW YORK PRICES Y_1

Table IX shows the correlation between the link relatives of the world-production of sugar and the link relatives of New York wholesale prices of granulated sugar, for various lags. The data used for computing these correlations are shown in Table III of Appendix III.

TABLE IX

COEFFICIENTS OF CORRELATION BETWEEN WORLD-PRODUCTION X_w, AND NEW YORK WHOLESALE PRICES OF SUGAR Y_1 FOR VARIOUS LAGS*

Series Correlated	Coefficient of Correlation (r)
Y_1 (1903) and X_w (1904)	+0.59
Y_1 (1903) and X_w (1903)	−0.85
Y_1 (1904) and X_w (1903)	+0.45

* The figures in the brackets indicate the *first* year of each of the correlated series under consideration. Thus, Y_1 (1903) means the average price for the fiscal year beginning July 1, 1903; X_w (1904) means the world-production for the fiscal year beginning July 1, 1904, etc.

It will be noticed that the *signs* of the correlation coefficients are the *reverse* of what might have been expected from Tables VI, VII, and VIII. The correlation between New York prices for any given year and world-production for the *following year* is *positive,* and the correlation between New York prices for any given year and production for the *same year* is *negative.* The positive coefficient of correlation, $r=+0.59$, between the price for any given year and the production for the following year, is the correlation for *supply.* The negative coefficient of correlation, $r=-0.85$, between the price for any given year and the production for the same year, is the correlation for *demand.* This is perhaps an illustration of the principle that, in a self-contained economy, the supply-demand relation approximates its theoretical simplicity.[31] Low production for any year brings about high prices in the same year; these high prices in turn call forth a higher production the following year.

[31] See pp. 101–3, 132–36.

2. EQUATIONS CONNECTING WORLD-PRODUCTION X_w AND PRICE Y_1

The "lines of best fit" giving the most probable supply and demand relationship between X_w and Y_1 are:

$$Y_1 = +1.6788X_w - 0.751 \tag{50}$$

for supply, and

$$Y_1 = -1.4888X_w + 2.554 \tag{51}$$

for demand, the origin for each curve being at 0,0.

Equation (50) means that, based on the experience for 1903–13, an increase (or decrease) of one *point* (or roughly 1 per cent) in the link relative of world-production for any given year is, *on the average*, associated with an increase (or decrease) of 1.68 *points* in the New York price for the preceding year. Another way of saying the same thing is that an increase (or decrease) of one *point* in the relative price for any given year is on the average associated with an increase (or decrease) of $1/1.6788 = 0.60$ of a *point* in the relative production for the following year. This is the law of the world-supply.

Equation (51) tells us that an increase (or decrease) of one *point* in the link relative of world-production (= consumption) for any given year is on the average associated with a *decrease* (or increase) of 1.49 *points* in the New York price *for the same year*, or that an increase (or decrease) of one *point* in the New York price for any given year is on the average associated with a *decrease* (or increase) of 0.67 of a point in the world-consumption for the same year. This is the law of the world-demand. The assumption is made that the world-production for any year is, on the average, approximately equal to the world-consumption for the same year.

3. A MOVING EQUILIBRIUM OF DEMAND AND SUPPLY

Figure 35 brings the supply curve, SS', into juxtaposition with the demand curve, DD', and shows the individual observations on which the curves are based. A circle indicates a demand observation and a cross indicates a supply observation. The

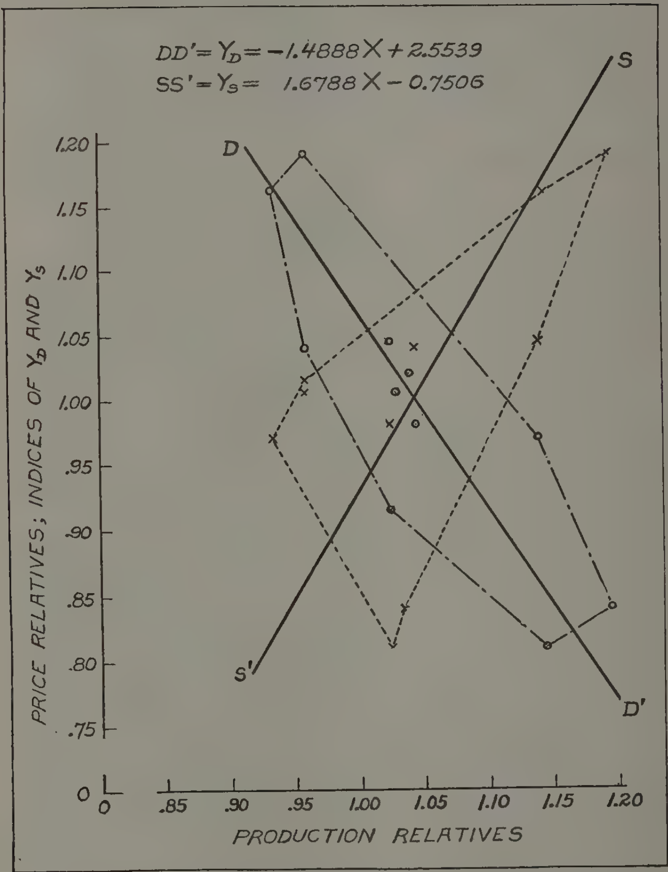

FIG. 35.—Probable moving equilibrium of world's sugar demand and supply. A circle indicates a demand observation, and a cross indicates a supply observation. The range covered by each set of observations is indicated by the areas enclosed by the dotted lines.

range covered by each set of observations is indicated by the areas enclosed by the dotted lines.

The two curves intersect at the point whose abscissa is the arithmetic mean of the quantity relatives (1.043) and whose ordinate is the arithmetic mean of the price relatives (1.001).[32] That is to say, during the period covered by this study (1903–13), the world-production of sugar for any one year was, *on the average,* 104.3 per cent of the production for the previous year, and the *corresponding* New York price of sugar was, *on the average,* 100.1 per cent of the price for the preceding year. These figures are assumed to give, respectively, the "normal" trend of production and the "normal" trend of prices for the period under consideration. Since the two curves pass through the point whose co-ordinates are these "normal" figures of production and price, the demand for the commodity and the supply of the commodity are in what Professor Moore has termed[33] "moving equilibrium" about the trends of production and prices. When, for example, the supply price relative of a given year was 1.001, the production relative of the following year was 1.043; and when the production relative of that year was 1.043, the demand price relative of the same year was 1.001.

4. ELASTICITIES OF WORLD-SUPPLY AND DEMAND FOR SUGAR

The elasticity of the world-supply is

$$e = +\frac{1}{1.6788}\,\frac{Y_1}{X_w}, \tag{52}$$

[32] This is true only approximately, because the correlation for demand is based on eleven pairs of observations as compared with ten pairs for supply, the difference being due, of course, to the loss of one pair of observations by "lagging" production behind prices by one year. In the correlation for demand the arithmetic means of the variables are $\overline{X}=1.047$, $\overline{Y}=1.006$. In the correlation for supply, the corresponding means are $\overline{X}=1.045$, $\overline{Y}=0.998$. It will be observed that these points lie very close to the point of intersection of the demand curve and the supply curve. When the small number of observations is taken into account, there is probably no significant difference between the corresponding co-ordinates of any two of the three points under consideration.

[33] *Quarterly Journal of Economics,* XXXIX (May, 1925), p. 370.

and the elasticity of the world-demand is

$$\eta = -\frac{1}{1.4888}\frac{Y_1}{X_w}. \tag{53}$$

At the point of intersection of the two curves, the elasticities of supply and demand are:

$$e = +0.57$$

and

$$\eta = -0.64 ,$$

respectively; or an increase in the price of 1 per cent will bring about an increase in production of 0.57 of 1 per cent and a decrease in consumption of 0.64 of 1 per cent.[34]

However, there is probably no statistically significant difference between the two coefficients. When the world-supply and demand curves are deduced from the data for 1890–1913 instead of from those for 1903–1913, the foregoing coefficients have the values of $e = +0.59$ and $\eta = -0.61$. We conclude, therefore, that the elasticity of the world-supply is numerically equal to the elasticity of the world-demand and that it is approximately of the order of 0.6. This is perhaps an illustration of the principle developed above,[35] that in a self-contained economy the elasticities of demand and supply *tend* to be numerically equal.

VII. INTERRELATIONS OF PRICES, PRODUCTION, AND IMPORTS

In the preceding correlations no attempt was made to measure the *net* relation between any two of the variables considered while the others are kept constant, except in so far as the use of the "line of best fit" may be said to do so indirectly and approximately. The correlation coefficients thus obtained are, therefore,

[34] When prices for the eight months beginning October 1 (the world's harvesting period) are substituted for the average annual prices used in the foregoing computations, the elasticities of supply and demand become +0.42 and −0.48, respectively.

[35] See pp. 101–3, 132–36.

gross, or *zero order* coefficients. Neither was there an attempt made to measure the *cumulative* influence upon any one variable of all the others. Thus, there was first determined the relation between domestic production and prices; then the relation between imports from insular possessions and prices; and finally, the relation between world-production and prices. No attempt was made, however, to measure the combined effect of all the three variables on prices. Do not these shortcomings invalidate our conclusions? Would not the application of the methods of partial and multiple correlation have led to different results? We must seek answers to these questions.

I. PARTIAL CORRELATIONS

Table X gives the net or partial coefficients of correlation between prices, production, and imports. All the variables considered heretofore are represented in this table, except Y_3, the average New York price for the six months beginning January 1. All the variables are link relatives for the eleven fiscal years, 1903–13. No correlations between "lagged" series are included in this table.

The net correlations have the same signs as the corresponding gross correlations, but are of lower magnitude.

The coefficient of correlation between domestic production and New York prices, when imports from insular possessions are kept constant, is between +0.35 and +0.42, depending upon the price series used. The gross correlation is between +0.49 and +0.54 (see Table VI). When both imports from insular possessions and world-production are kept constant, the correlation drops to between +0.14 and +0.26.

The coefficient of correlation between imports from insular possessions and New York prices, when domestic production is kept constant, is between +0.67 and +0.70, depending upon the price series used. The gross correlation is between +0.72 and +0.75 (see Table VII). When both domestic production and world-production are kept constant, the correlation is lowered to between +0.41 and +0.46.

The correlation between domestic production plus imports from insular possessions and New York prices, when world-pro-

TABLE X

NET COEFFICIENTS OF CORRELATION BETWEEN PRODUCTION, IMPORTS, AND PRICES OF SUGAR, FOR THE PERIOD 1903–13

Variables Correlated	Variables Held Constant	Net Correlation	
		Symbol	Numerial Value
X_1 and Y_1	X_2	$r_{x_1 y_1 \cdot x_2}$	+0.35
X_1 and Y_2	X_2	$r_{x_1 y_2 \cdot x_2}$	+0.42
X_1 and Y_1	X_2 and X_w	$r_{x_1 y_1 \cdot x_2 x_w}$	+0.14
X_1 and Y_2	X_2 and X_w	$r_{x_1 y_2 \cdot x_2 x_w}$	+0.26
X_2 and Y_1	X_1	$r_{x_2 y_1 \cdot x_1}$	+0.70
X_2 and Y_2	X_1	$r_{x_2 y_2 \cdot x_1}$	+0.67
X_2 and Y_1	X_1 and X_w	$r_{x_2 y_1 \cdot x_1 x_w}$	+0.46
X_2 and Y_2	X_1 and X_w	$r_{x_2 y_2 \cdot x_1 x_w}$	+0.41
X_3 and Y_1	X_w	$r_{x_3 y_1 \cdot x_w}$	+0.45
X_3 and Y_2	X_w	$r_{x_3 y_2 \cdot x_w}$	+0.49
X_w and Y_1	X_1	$r_{x_w y_1 \cdot x_1}$	−0.81
X_w and Y_2	X_1	$r_{x_w y_2 \cdot x_1}$	−0.78
X_w and Y_1	X_2	$r_{x_w y_1 \cdot x_2}$	−0.73
X_w and Y_2	X_2	$r_{x_w y_2 \cdot x_2}$	−0.70
X_w and Y_1	X_3	$r_{x_w y_1 \cdot x_3}$	−0.70
X_w and Y_2	X_3	$r_{x_w y_2 \cdot x_3}$	−0.66
X_w and X_1	X_2	$r_{x_w x_1 \cdot x_2}$	−0.35
X_w and X_2	X_1	$r_{x_w x_2 \cdot x_1}$	−0.59

duction is kept constant, is between +0.45 and +0.49, depending upon the price series used. The gross correlation between the two variables is +0.76 (see Table VIII).

The correlation between world-production and New York prices when domestic production is kept constant is between —0.78 and —0.81, depending upon the price series used. The gross correlation is —0.85 (see Table IX). The correlation drops to between —0.70 and —0.73, when imports from insular possessions are kept constant; and to between —0.66 and —0.70, when both domestic production and imports from insular possessions are kept constant.

The figures of Table X thus show clearly how relatively unimportant domestic production was as a "price determinant" during 1903–13. When both world-production and imports from insular possession are kept constant, the correlation between United States production and New York price becomes so low as to be of questionable statistical significance.

The correlations of Table X also bring into clear relief the predominant effect of world-production on New York prices. The correlation between these two variables is not less than —0.7 (approximately), even if the supply of sugar from domestic and insular sources is kept constant. This, however, is a *demand* relationship (see Table IX).

2. MULTIPLE CORRELATIONS

The foregoing net correlations suggest the desirability of completing the analysis by determining the cumulative effect upon any one variable of some or all of the others. However, before proceeding to the consideration of the numerical results, it is advisable to state briefly some of the more important relations of the calculus of multiple correlation.

Although some phases of the theory of multiple correlation may also be applied to the study of non-linear relationships, the theory has progressed farthest and has had its widest application in the study of the correlation among three or more variables which are linearly dependent. This means that the equation connecting the variables is the linear regression surface

$$Z_1 = a + b_{12\cdot 34\cdots n}Z_2 + b_{13\cdot 24\cdots n}Z_3 + \cdots + b_{1n\cdot 2\cdots (n-1)}Z_n \qquad (54)$$

where $Z_1, Z_2, \ldots Z_n$ are the variables under consideration. The statistical problem is to determine the parameters $b_{12\cdot3\cdots n}$, $b_{13\cdot24\cdots n}$, ... , $b_{1n\cdot2\cdots n-1}$, and a so as to obtain the "best" estimates of Z_1, corresponding to any assigned values of $Z_2, Z_3, \ldots, Z_n$.

It is convenient for theoretical purposes to work with the deviations of the Z's from their own means, divided by their own standard deviations.

Putting

$$z_1 = \frac{Z_1 - M_1}{\sigma_1}, \qquad z_2 = \frac{Z_2 - M_2}{\sigma_2}, \text{ etc.} \ldots \tag{55}$$

and

$$\begin{aligned} b_{12\cdot34\cdots n} &= \beta_{12\cdot34\cdots n}\frac{\sigma_1}{\sigma_2} \\ b_{13\cdot24\cdots n} &= \beta_{13\cdot24\cdots n}\frac{\sigma_1}{\sigma_3} \\ b_{1n\cdot23\cdots(n-1)} &= \beta_{1n\cdot23\cdots n-1}\frac{\sigma_1}{\sigma_n} \end{aligned} \tag{56}$$

equation (54) becomes

$$z_1 = \beta_{12\cdot34\cdots n}z_2 + \beta_{13\cdot24\cdots n}z_3 + \cdots + \beta_{1n\cdot23\cdots n-1}z_n \,. \tag{57}$$

The β's in this equation are called "partial regression coefficients," or "path coefficients," a term due to Sewall Wright.[36] Any partial regression coefficient—say, $\beta_{12\cdot3\ldots n}$—has the two following important properties:[37]

1. It is the regression of that part of z_1 which is independent of $z_3, z_4, \ldots, z_n$ upon that part of z_2 which is independent of $z_3, z_4, \ldots, z_n$.

2. It is the weight or multiplying factor of z_2 when $z_3, z_4, \ldots, z_n$ are used to estimate z_1.

[36] Sewall Wright, "Correlation and Causation," *Journal of Agricultural Research*, XX, No. 7 (January 3, 1921), 561–62.

[37] Truman Kelley, *Statistical Method* (1923), p. 285.

Adopting the least square criterion, we may determine the coefficients in (57) so that

$$f=\Sigma(z_1-\beta_{12\cdot34\cdots n}z_2-\beta_{13\cdot24\cdots n}z_3-\cdots-\beta_{1n\cdot23\cdots n-1}z_n)^2 \tag{58}$$

shall be a minimum. The answer, which gives the regression of z_1 on the other variables, may be written in several forms. One of these, which is convenient for theoretical purposes, makes use of a determinant involving the gross correlations between the variables.[38]

As soon as the β's have been determined, the b's of (54) may be found from (56), and the term a of the same equation from

$$a=M_1-b_{12\cdot34\cdots n}M_2-b_{13\cdot24\cdots n}M_3-\cdots-b_{1n\cdot23\cdots n-1}M_n\,. \tag{59}$$

Equation (58) is not ordinarily the most convenient form to use for numerical computations. However, it was necessary to introduce it for the light that its coefficients—the B's—throw on the problem of measuring the degree to which the dependent variable is "determined" by any of the independent variables.

Having outlined a method for determining the parameters of equation (54), we must next show how to measure the agreement between the fitted regression surface and the observations. The two measures that are commonly used for this purpose are the standard error and the coefficient of multiple correlation. The standard error of estimate, $S_{1\cdot23\ldots n}$, is a *concrete* number measuring the dispersion (scatter) of the observed values of Z_1 from its corresponding computed values on the hyperplane (54). It is an extension of the standard error of estimate with two variables (p. 43). It is defined as the root-mean-square of the deviations; that is,

$$S^2_{1\cdot23\cdots n}=\frac{1}{N}\Sigma(\text{observed } Z_1-\text{computed } Z_1)^2\,. \tag{60}$$

The multiple correlation coefficient, $R_{1\cdot23\ldots n}$, is an *abstract* number. It is the simple coefficient of correlation between

[38] See, for example, Henry L. Rietz, editor, *Handbook of Mathematical Statistics* (1924), p. 141.

the observed values of Z_1 and its corresponding estimated values calculated from the linear function (54).

The standard error and the coefficient of multiple correlation are connected by the formula[39]

$$R^2_{1\cdot 23\cdots n} = 1 - \frac{S^2_{1\cdot 23\cdots n}}{\sigma^2_1} \tag{61}$$

where σ_1 is the standard deviation of Z_1.

From this it follows that

$$S^2_{1\cdot 23\cdots n} = \sigma^2_1(1 - R^2_{1\cdot 23\cdots n}) . \tag{62}$$

This is an extension of the standard error of estimate with two variables (p. 43). It shows how the standard error decreases as the multiple correlation coefficient increases. When the latter is zero, the standard error is equal to the standard deviation; the fitting of the hyperplane (54) has served no useful purpose. When the correlation is perfect, the standard error is zero; there is no scatter; all the points fall on the hyperplane. The higher the correlation, the better the fit.

But it can be shown that[40]

$$R^2_{1\cdot 234\cdots n} = \beta_{12\cdot 345\cdots n}\, r_{12} + \beta_{13\cdot 245\cdots n}\, r_{13} + \cdots\cdots + \beta_{1n\cdot 234\cdots n-1} r_{1n} \cdot \tag{63}$$

[39] Frederick C. Mills, *Statistical Methods Applied to Economics and Business* (1924), p. 497.

[40] The proof is as follows: Let r_{12}, r_{13}, etc., be respectively the coefficients of correlation between variables 1 and 2, 1 and 3, etc. Let Δ stand for the determinant:

$$\Delta = \begin{vmatrix} 1 & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{12} & 1 & r_{23} & \cdots & r_{2n} \\ r_{13} & r_{23} & 1 & \cdots & r_{3n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ r_{1n} & r_{2n} & r_{3n} & \cdots & 1 \end{vmatrix}$$

and let Δ_{11}, Δ_{12}, etc., stand for the minors obtained by deleting the first row and

Substituting this in (62), we get

$$S^2_{1\cdot 234\cdots n} = \sigma_1^2(1 - \beta_{12\cdot 3\cdots n}r_{12} - \beta_{13\cdot 2\cdots n}r_{13} - \cdots - \beta_{1n\cdot 2\cdots n-1}r_{1n}). \quad (64)$$

This is one of the most useful formulas in the correlational calculus. The terms in βr, which have been called "*coefficients of determination*,"[41] give the *proportion* of the total variance (σ_1) of Z_1 which is due to each variable on the right hand side of (54). They thus measure quantitatively the importance of Z_2, Z_3, etc., as "causes" of Z_1. The difference between unity and the sum of the "coefficients of determination" measures the importance of factors which have not been taken into account, assuming that

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the first column, the first row and the second column, etc. With this notation, the β's are given by the well-known formula (see Henry L. Rietz, editor, *Handbook of Mathematical Statistics*, p. 141):

$$\begin{aligned} \beta_{12\cdot 34\cdots n} &= \Delta_{12}/\Delta_{11}, \\ \beta_{13\cdot 24\cdots n} &= \Delta_{13}/\Delta_{11}, \\ \beta_{14\cdot 235\cdots n} &= \Delta_{14}/\Delta_{11}, \\ \beta_{1p\cdot 23\cdots(p)\cdots n} &= (-1)^p\Delta_{1p}/\Delta_{11}, \\ &\text{etc.} \end{aligned} \quad \text{(i)}$$

and

$$R^2_{1\cdot 23\cdots n} = 1 - \frac{\Delta}{\Delta_{11}} \quad \text{(ii)}$$

a formula due to Pearson.

But, by the theory of determinants,

$$\Delta = \Delta_{11} - \Delta_{12}r_{12} + \Delta_{13}r_{13} - \cdots\cdots - (-1)^n\Delta_{1n}r_{1n}. \quad \text{(iii)}$$

Dividing both sides by Δ_{11}, we have

$$\frac{\Delta}{\Delta_{11}} = 1 - \frac{\Delta_{12}}{\Delta_{11}}r_{12} + \frac{\Delta_{13}}{\Delta_{11}}r_{13} - \cdots\cdots - (-1)^n\frac{\Delta_{1n}}{\Delta_{11}}r_{1n}. \quad \text{(iv)}$$

Substituting in (ii),

$$R^2_{1\cdot 23\cdots n} = \frac{\Delta_{12}}{\Delta_{11}}r_{12} - \frac{\Delta_{13}}{\Delta_{11}}r_{13} + \cdots\cdots + (-1)^n\frac{\Delta_{1n}}{\Delta_{11}}r_{1n}. \quad \text{(v)}$$

But, by (i), $\Delta_{12}/\Delta_{11} = \beta_{12\cdot 345\cdots n}$, $\Delta_{13}/\Delta_{11} = \beta_{13\cdot 245\cdots n}$, etc. Substituting in (ii), we get equation (63).

[41] See note 36, p. 172.

the statistics used are free from errors of observation, and that the regression of Z_1 upon Z_2, Z_3, etc., is linear.

We are now ready for a consideration of the numerical results obtained by the method of multiple correlation.

The regression of Y_1 on X_1, X_2, and X_w, is

$$Y_1 = 1.5081 + 0.07331X_1 + 0.2905X_2 - 0.8585X_w \tag{65}$$

the origin being at 0,0.

This equation enables us to compute the average annual New York price of sugar corresponding to any combination of domestic production, imports from insular possessions, and world-production.[42]

The standard error is $S_{Y_1 \cdot X_1 X_2 X_w} = 0.0505$. This means that the computed price relatives do not differ by more than ± 0.0505 units from the observed price relatives in approximately two-thirds of the cases. The standard deviation of the price relatives is $\sigma_{Y_1} = 0.1113$.

The multiple correlation coefficient is $R_{Y_1 \cdot X_1 X_2 X_w} = 0.891$. The square of this coefficient, or 0.794, measures (see equation [62]) the proportion by which the square of the error of estimating Y_1 has been reduced through the use of equation (65). If this equation were not available, we could give only one and the same estimate of Y_1 (its arithmetic mean) for any combination of X_1, X_2, and X_w; and the confidence which we would be justified in placing in such an estimate would be measured by the standard deviation. By using equation (65) to estimate Y_1, we are able to reduce the square of the standard deviation (called *variance*) by 79.4 per cent. To this reduction, world-production contributed 52.1 per cent; imports from insular possessions, 23.7 per cent; and domestic production, only 3.6 per cent.[43] The remaining 20.6 per cent is due either to factors which have not

[42] See pp. 39–40 for an illustration of method when only two variables are involved.

[43] These figures were computed from (63). The β's may be deduced from the b's by (56); the r's, from Tables VI, VII, and VIII, and the σ's from the fol-

been taken into account, or to inaccuracies and deficiencies of the data, or to the impossibility of adequately representing the observations by a linear function. These figures, which are the numerical values of the "coefficients of determination," thus measure the importance of the several factors as "price determinants."

In this connection it must be emphasized that the foregoing figures are measures of *net* contributions. Thus, when it is said that world-production is responsible for 52.1 per cent of the variance of Y_1 the statement is to be understood as meaning "that part of the world-production which is independent of changes in United States production and imports from insular possessions." A corresponding interpretation must be given to the effects of the other factors. If it be argued that world-production cannot make a contribution to the variance of price, which is essentially independent of United States production and imports from insular possessions, the answer is that under these circumstances we have no right to use the linear function (54).

Similar results are obtained when domestic production and imports from insular possessions are treated as one variable. We have then

$$Y_1 = 1.5367 + 0.3700X_3 - 0.8912X_w \tag{66}$$

the origin being at 0,0. The standard error is $S_{Y_1 \cdot x_3 x_w} = 0.0470$ (in link relative units), and the coefficient of multiple correlation

lowing table, which also contains the means of the variables. In the computations, the r's were carried out to more places than are given in Tables VI, VII, and VIII.

Variable	Mean	Standard Deviation
Y_1	0.9978	0.1113
X_1	1.0631	0.1119
X_2	1.0637	0.1215
X_3	1.0612	0.0968
X_w	1.0452	0.0790

is $R_{Y_1 \cdot X_3 X_w} = 0.885$. The two factors X_3 and X_w account for 78.5 per cent of the total variance of Y_1. World-production accounts for 54.1 per cent of the variance, and domestic production, with imports from insular possessions, for 24.4 per cent. We have seen that the corresponding percentages deduced from (65) are 52.1 and 27.3, respectively.

3. LIMITATIONS OF THE MULTIPLE CORRELATION APPROACH

If our purpose were only to develop a method of estimating the probable price corresponding to a given combination of factors such as X_1, X_2, and X_w, which are assumed to be known accurately, the accepted method of multiple regression, such as is illustrated by equation (65) or (66), might be used to advantage. It gives a measure of the accuracy of the estimate and of the importance of each of the factors and suggests ways of improving the estimate, should this be necessary. But we wish our equation to express the best general relationship connecting all of the variables in question, so as to enable us to deduce unequivocal values for the elasticities of supply and demand. For this purpose the accepted multiple-regression approach, as typified by equation (65), is generally liable to be misleading.

To illustrate the difficulties raised by such equations as (65) or (66), let us rewrite (65) so as to bring X_1 to the left-hand side:

$$X_1 = -20.571 + 13.64Y_1 - 3.962X_2 + 11.71X_w \tag{65a}$$

and compare it with the true regression[44] of X_1 on Y_1, X_2, and X_w:

$$X_1 = 1.1876 + 0.2650Y_1 + 0.003535X_2 - 0.3757X_w \tag{67}$$

the origin being at 0,0.

[44] For equation (67), the standard error is 0.0967, and the coefficient of multiple correlation is 0.512. The use of this equation effects a reduction of 26.2 per cent in the variance of X_1. To this reduction, Y_1 contributes 13.0 per cent; X_2, 0.1 per cent; and X_w, 13.1 per cent.

Let us also rewrite (66) as follows:

$$X_3 = -4.1532 + 2.7027 Y_1 + 2.4086 X_w \tag{66a}$$

and compare it with the true regression[45] of X_3 on Y_1 and X_w:

$$X_3 = 0.7150 + 0.5447 Y_1 - 0.1888 X_w \tag{68}$$

the origin being at 0,0.

The differences between the corresponding equations are too striking to require pointing out. They lead to correspondingly striking differences in the coefficients of elasticity of supply (or demand) derived from them. Thus the elasticity of the domestic supply when the other factors are kept constant, or the *partial elasticity of supply*[46] for X_1 is

$$e_{x_1 y_1 \cdot x_2 x_w} = \frac{\partial X_1}{\partial Y_1} \cdot \frac{Y_1}{X_1} = 13.64 \frac{Y_1}{X_1} \tag{69}$$

from (65) or (65a); and

$$e_{x_1 y_1 \cdot x_2 x_w} = \frac{\partial X_1}{\partial Y_1} \cdot \frac{Y_1}{X_1} = 0.2650 \frac{Y_1}{X_1} \tag{70}$$

from (67). At the point on the curves whose co-ordinates are the arithmetic means of X_1 and Y_1 (see note 43, pp. 176–77), these coefficients have the values of 12.8 and 0.25, respectively. Similar discrepancies occur in the coefficients for X_3 and X_w. Thus, the elasticity of the supply from our insular possessions, when world-production and United States production are kept constant, is 3.23 by (65) or (65a), and -70.3 by (67). Similarly the elasticity of the world-demand when United States pro-

[45] For equation (68), the standard error is 0.0626, and the coefficient of multiple correlation is 0.762. The use of this equation effects a reduction of 58.1 per cent in the variance of X_3. To this reduction, Y_1 contributes 47.5 per cent, and X_w, 10.6 per cent.

[46] These concepts of *partial* elasticities of demand and supply are due to Professor Henry L. Moore. See his papers: "Partial Elasticity of Demand," *Quarterly Journal of Economics* (May, 1926); and "A Theory of Economic Oscillations," *ibid.* (November, 1926).

duction and imports from insular possessions are kept constant is either -1.11 or $+0.67$, depending upon whether (65) or (67) is used.

Turning to (66a) and (68), we find similar differences. Thus, the partial elasticity of supply for X_3 is 2.54 by (66a), and only 0.51 by (68). The corresponding figures for the partial elasticity of the world-demand (X_w) are -1.07, and $+0.27$.

These anomalous results are due in part to the same causes which produced the divergent results in the study of demand when only two variables were considered:[47] the arbitrary selection of one of the variables as the dependent variable, and the assumption that an observed point fails to fall on the plane because of an "error" or deviation in this variable alone, the set of independent variables being allowed no deviation. This gives each independent variable an infinite weight as compared with the dependent variable, and the result is that with the same method of fitting (least squares) and the same set of variables we get one curve if we treat Y_1 as the dependent variable, as in (65) or (66), and a quite different curve if we treat either X_1 or X_3 as the dependent variable, as in (67) or (68).

The method of overcoming this difficulty is similar to that used in the two-variable problem (see pp. 35–39). It directs us so to fit the hyperplane (54) as to take into account "errors" or deviations in all the variables considered. This method cannot, however, be safely employed in the four-variable problem immediately before us, for the following reasons:

1. The number of observations is very small. To fit an equation with four constants to eleven points in space is to build an elaborate mathematical superstructure on a somewhat shaky statistical foundation; and the results obtained are, therefore, apt to be of little or no statistical significance. Though this criticism is equally applicable to all other methods of curve-fitting, it has added significance when the method employed is quite cumbrous, as is the one under consideration.

[47] See pp. 35–39.

2. The method assumes a knowledge of the weights to be applied to the several variables and raises the question of the effect of the units in which the several variables are expressed on the fit that is obtained. And in the problem before us there is no simple, unequivocal criterion for the determination of proper weights.

In the two-variable problem, or in the fitting of the straight line, we assumed that the trend ratio or the link relative is a good unit in which to measure the variables (although the dispersion of these ratios or relatives varies from series to series), and that it is more reasonable to give both variables the same weight than to give one of them an infinite weight as compared with the other. The method of curve-fitting which is prescribed by these assumptions is that which makes the sum of the squares of the perpendicular deviations of the points from the line a minimum.[48]

Accordingly, we adopted this criterion of fit, and obtained fairly consistent results, as indicated by the practical constancy of the elasticity of demand deduced from the various lines which were fitted by this method.

In the four-variable problem immediately before us, however, these assumptions cannot safely be made, for the reason that world-production is so much more important as a "price determinator" (see p. 176) that a slight change in the weights assigned to the several variables is apt to have a comparatively large effect on the coefficients of X_1 and X_2. Two illustrations will suffice.

If we treat Y_1, X_3, and X_w as equally accurate or equally inaccurate and fit a plane to these three[49] variables by minimizing

[48] Had a different assumption been made regarding the relative importance of the two variables and the units in which they are measured, a function other than the sum of the squares of the normal deviations would have had to be minimized.

[49] This method of fitting becomes more and more cumbersome as the number of variables is increased. It was therefore decided to confine this illustration to these three variables, and not to treat X_1 and X_2 separately.

the sum of the squares of the perpendiculars from the system of points upon the plane, we get the following result:

$$Y_1 = 2.2828 + 0.1243 X_3 - 1.3556 X_w \tag{71}$$

the origin being at 0,0.

Judged by its standard error, which is 0.03438 (link relative units), this equation fits the observations very well. But when it is used to compute the partial elasticity of supply for X_3 and the partial elasticity of demand for X_w, we obtain:

$$e_{X_3 Y_1 \cdot X_w} = \frac{\partial X_3}{\partial Y_1} \cdot \frac{Y_1}{X_3} = + \frac{1}{0.1243} \frac{Y_1}{X_3} \tag{72}$$

and

$$\eta_{X_w Y_1 \cdot X_3} = \frac{\partial X_w}{\partial Y_1} \cdot \frac{Y_1}{X_w} = - \frac{1}{1.3556} \frac{Y_1}{X_w} \tag{73}$$

which, at the points represented by the arithmetic means of the variables in question (see note 43, page 177), have the values of 7.56 and —0.70, respectively. The value of —0.70 for the elasticity of the world-demand is not significantly different from the value of —0.64 obtained by using the straight line connecting X_w and Y_1 (see page 168). However, the value of 7.56 for the elasticity of the supply from domestic and insular sources is much too large, and is probably due to the particular scale and method of fitting employed.

Let us therefore adopt a somewhat different method of procedure. Let us first express the link relatives Y_1, X_3, and X_w in terms of their respective standard deviations and then fit a plane to these adjusted relatives by the same method that was used to fit (71). The resulting equation, reconverted into the original or unadjusted relatives, is:

$$Y_1 = 1.8741 + 0.2787 X_3 - 1.1213 X_w \tag{74}$$

the origin being at 0,0.

A comparison of (71) and (74) shows that the effect of the modification of the units in which the variables were expressed has been to raise the coefficient of X_3 by more than 100 per cent and to lower the coefficient of X_w by only 10 per cent (approximately).

The partial elasticity of supply for X_3 and the partial elasticity of demand for X_w deduced from this equation are:

$$e_{x_3 y_1 \cdot x_w} = + \frac{1}{0.2787} \cdot \frac{Y_1}{X_3} \tag{75}$$

and

$$\eta_{x_w y_1 \cdot x_3} = - \frac{1}{1.1213} \cdot \frac{Y_1}{X_w} \tag{76}$$

which, at the points represented by the arithmetic means of the variables in question (see note 43, page 177), have the values of +3.37 and —0.85, respectively. These must be compared with the corresponding values deduced from (71), which are 7.56 and —0.70.

Recourse to the method used in (74) has thus resulted in a *large* reduction in the numerical value of the elasticity of supply from non-dutiable sources when world-production is kept constant, and in a *small* reduction in the numerical value of the elasticity of world-demand when the combined supply from United States production and imports from insular possessions is kept constant. It is thus seen that a change in the weights assigned to the several variables (which is involved in the use of the standard deviation as a unit of measure) may have an important effect on the coefficients of the variables and on related functions.

Both equations (71) and (74) seem to indicate that the world-demand for sugar (when United States production and imports from insular possessions is kept constant) is inelastic, while the supply which is derived from United States production and imports from insular possessions (when world production is kept constant) is elastic. The first conclusion is in substantial

agreement with that yielded by the simple, straight line approach (see p. 168). The second is not (see p. 162). This raises the question of the reliability of the two sets of determinations. We have stated our reasons for believing that in the problem before us the results yielded by the simpler methods are more likely to be correct: there is not enough of a statistical foundation to justify a multiple correlation approach except as an illustration of method; and the different methods of fitting the equation are apt to lead to different conclusions as to the elasticity of supply. But the more complex methods do, nevertheless, cast some doubt on the earlier conclusion that the supply of sugar from non-dutiable sources is inelastic. The problem, therefore, requires further examination.

The entire difficulty is due to our effort to find the elasticity of supply from non-dutiable sources while world-production is kept constant. From an economic point of view it may well be argued that world-production should not be kept constant, since it is immaterial whether it is a world-production or any other factor that brings about fluctuations in prices, so long as the latter, in turn, bring about fluctuations in United States production and non-dutiable imports, and since, furthermore, world-production *includes* United States production and imports. Whatever merits these arguments may have, they do not dispose of the methodological difficulty involved in any attempt to remove the influence of any one variable upon any other variable or group of variables. If the accepted methods of dealing with this difficulty are either too cumbersome or otherwise unavailable for the purpose in view, may not a modification of them be more suitable for our purposes? Let us see.

The "disturbing factor" for our purposes is X_w. The correlation between Y_1 and X_3 when X_w is kept constant is, by Table X, $r_{X_3 Y_1 \cdot X_3} = +0.45$. This *partial* correlation is equal to the *simple* correlation between the two series Y_1 and X_3, when the contribution of X_w has been removed from each of them. This can be done as follows:

1. Find the linear regression of Y_1 on X_w and the linear regression of X_3 on X_w. They are:

$$Y_1 = 2.2559 - 1.2037 X_w \tag{77}$$

and

$$X_3 = 1.9439 - 0.8445 X_w \tag{78}$$

respectively. These equations enable us to compute estimates of Y_1 and X_3 from a knowledge of X_w.

2. For each observed Y_1 compute a probable Y_1 from (77), and for each observed X_3 compute a probable X_3 from (78). (See Appendix III, Tables I A and III, for the observed values of the three variables.) The values computed from (77) measure the degree to which sugar prices depend on world-production. Similarly, the values computed from (78) measure the degree to which domestic production and imports depend on world-production.

3. Subtract the computed Y_1's from the observed Y_1's, and the computed X_3's from the observed X_3's. The first set of residuals represents that part of Y_1 which cannot be estimated from a knowledge of X_w, or that part which is independent of X_w. The second set of residuals represents that part of X_3 which cannot be estimated from X_w, or that part which is independent of X_w.

4. Plot a scatter diagram of the two series of residuals in order to see whether the regression is linear. The simple coefficient of correlation between the two sets of residuals is, as was stated before, the partial coefficient of correlation between Y_1 and X_3 for a constant X_w. The regression of the price residuals on the quantity residuals has a slope of $+0.3700$, and is also equal to the coefficient of X_3 in equation (66). The regression of the quantity residuals on the price residuals has a slope of $+0.5447$, and is also equal to the coefficient of Y_1 in equation (68).[50] The use of the first regression would lead to the conclusion that the

[50] For proof, see Kelley, *op. cit.*, pp. 284–87.

supply from domestic and non-dutiable sources is elastic (2.54); the use of the second regression would lead to the conclusion that the supply is inelastic (0.51).

5. Instead of using one of these lines of regression, deduce the "line of mutual regression," or the "line of best fit," which is obtained by minimizing the sum of the squares of the normal deviations. This regression may be written

$$(Y_1 - \bar{Y}_1) = +1.4753(X_3 - \bar{X}_3) \tag{79}$$

where $\bar{Y}_1$ and $\bar{X}_3$ stand for the arithmetic means of Y_1 and X_3, respectively; and it leads to a coefficient of elasticity of supply of +0.68. This result agrees fairly well with those reached by the simpler methods of section V of this chapter (see pp. 161–62), thus reinforcing the conclusion reached therein that the supply of X_3 is inelastic.

The foregoing survey of the interrelations of prices, production, and imports shows that the ordinary method of multiple correlation, excellent as it may be for some purposes, is not a safe method for the purpose of finding elasticities of demand and supply. It is possible to modify this method so as to make it a better tool for our purposes, but the limitations of our data preclude much statistical experimentation in this direction. However, such results as we have obtained by the more complex methods do not disprove those reached by the simpler methods of the previous sections of this chapter. The elasticities of supply and demand obtained by the simpler methods of this chapter are considered therefore, the most probable values that can be deduced from the data at hand.

VIII. SUMMARY AND CONCLUSIONS RELATING TO SUPPLY

In order to bring the relation between the various subjects treated in this chapter into a rough unity, it may be convenient to summarize the various facts, inferences, and suggestions which have been detailed.

1. The results of a method cannot be separated from the method itself. The many reservations and limitations which attach to a given result and the fine differences which distinguish it from related results cannot be understood except in terms of the operations by which it was derived. Thus *the concept of elasticity of supply (or demand) is synonymous with the set of operations by which it is determined.* If we keep this in mind, and if we recall that the results relating to supply are based on the data for the "normal" period of 1903–13, during which there was no change in the duty on raw sugar; and that, therefore, they may not hold today, it will not be necessary to repeat all the reservations with which the various conclusions must be accepted.

2. The elasticities of the various supplies investigated in the foregoing pages vary according to the price series that are used. The elasticity of the domestic supply is between $+0.62$ and $+0.95$; the elasticity of the imports from non-dutiable sources is between $+0.72$ and $+1.05$; and the elasticity of the combined supply from domestic and non-dutiable sources is between $+0.56$ and $+0.78$. If we work with average annual prices the figures show that an increase of 1.0 per cent in the average price for any year is associated with an increase of *less* than 1.0 per cent in the supplies from domestic and other non-dutiable sources *for the same year,* whether they be taken severally or jointly. The coefficients are all derived from "lines of best fit," and this must be taken into account in comparing them with corresponding coefficients obtained by other methods.

3. The elasticity of the supply (e) from domestic and non-dutiable sources is numerically greater than the elasticity of the United States demand, which, it will be recalled, is $\eta = -0.5$, i.e., the supply from domestic and non-dutiable sources is relatively more elastic than the United States demand, although both are inelastic, i.e., both are numerically less than 1.0.

4. The world's supply of sugar is also inelastic, the coefficient of elasticity of supply, e_w, being $+0.57$—or, say, $+0.6$.

That is to say, a 1 per cent increase in the New York price for any year will call forth an increase of only 0.6 of 1 per cent in the world's production for the following year.

5. The world's demand for sugar is also inelastic, the elasticity of demand, η_w, being —0.64, or, say, —0.6, which means that a 1 per cent increase in the New York price for any year will reduce world-consumption in the same year by only 0.6 of 1 per cent. The numerical equality between e_w and η_w, which is to be expected in a self-contained economy only under some rather extreme assumptions, thus tends to be approximated in the world sugar market.

6. The foregoing values of the elasticities of supply and demand are valid only for the "normal conditions" defined in the text. A change in the conditions is accompanied by a change in the values of e and η. The amount of change in these coefficients corresponding to any change in price can easily be computed from the equations given in the text.

7. The factors considered in the study of supply account for 79.4 per cent of the variance (σ^2) of average annual prices. World-production accounts for 52.1 per cent; imports from insular possessions, for 23.7 per cent; and domestic production, for only 3.6 per cent.

8. The laws of supply and demand discussed in this study are all short-time laws:

> In the more fundamental questions which relate to long periods, the matter is . . . . more complex. For the ultimate output corresponding to an unconditional demand at even current prices would be theoretically infinite; and therefore the elasticity of supply of a commodity which conforms to the law of Increasing Return, or even to that of Constant Return, is theoretically infinite for long periods.[51]

We may, however, express short-time supply and demand curves in terms of the absolute quantities and prices for any one year and thus obtain what Professor Moore has termed "a moving equilibrium of demand and supply." For example, by ex-

[51] Marshall, *op. cit.*, pp. 456, 457.

pressing the world's supply and demand equations (see Fig. 28) for each year from 1903 to 1913 in terms of absolute quantities and prices,[52] we can show the "moving equilibrium," or the shift of the supply and demand curves from year to year as a result of dynamic changes. The graphical representation of this moving equilibrium calls for a diagram in three dimensions—the first for price, the second for quantities, and the third for *time*. The diagram will resemble a long, inclined, slightly undulating wire, around which are fastened, at their centers, pairs of pins roughly perpendicular to each other. The long wire represents the ratios between the computed trends of prices and the corresponding trends of the quantities (or the ratios between "normal" prices, and the corresponding "normal" quantities). It is about this long line—the trend of the ratios—that the pairs of pins, the short-time supply and demand curves, fluctuate. (See equations [5], p. 41; [8], p. 62; [11], p. 74; and [15], p. 85.)

9. The laws of supply derived in this study are *dynamic* laws; they describe in summary form the "routine of change" of important economic phenomena. They are the dynamic laws of supply in the simple form (link relatives). They are quite different from the static law of supply, which "may be only approached, but never realized, in inductive investigations."

10. Of the many questions that are raised by this study, two in particular deserve attention: (1) How does it come about that United States production of sugar is high when world-production is low, and vice versa? (The correlation between United States production [X_1] and world-production [X_w] is $r = -0.49$, and the correlation between United States production plus imports from insular possessions [X_3] and world-production [X_w] is $r = -0.69$.) (2) What is the explanation of the high negative correlation between United States production plus imports from insular possessions (X_3) for any one *fiscal* year, and the prices for the six-month period January to June (Y_3), *which begins*

[52] See pp. 40–42, 61–62, 74–76, and 85–87 for method.

half a year later? (The correlation between X_3 [1902] and Y_3 [1904] is $r = -0.94$. See Table VIII.)

11. The object of this study has been to make an exploration into the difficult and interesting field of statistical economics, not to develop methods of forecasting the price or the production of sugar. We venture to hope, however, that the results reached in this study will not be without interest to the trade.

In the following chapter the results reached in the foregoing pages will be used to estimate the effect of the tariff on sugar prices.

PART III. APPLICATIONS TO THE TARIFF

CHAPTER VI

COST OF PRODUCTION, SUPPLY AND DEMAND, AND THE TARIFF

I. THE PROBLEM: EFFECT OF A TAX ON THE PRICE OF A COMMODITY

Suppose the tariff on an important commodity—say sugar—is increased by one cent per pound. By how much will the tax raise the price paid by the consumer? What will be its effect on consumption? On domestic production? On imports? On the profits of the sugar industry?

These are the important questions on which depend the economic wisdom and social desirability of modifying the present tariff rate. They are of recurrent interest at nearly every presidential election. One would expect, therefore, that professional economists would be in a position to provide reliable *quantitative* answers to these questions. But not only have American economists been found wanting in this respect but they have, with a few glorious exceptions, even failed to develop the theory and technique without which no quantitative answers are possible.

II. POPULAR SOLUTIONS

This condition is to be deplored, because the methods which are commonly employed in attacking these problems are invalid. As an example, take the question of the effect of the tariff on only one factor—the price of the commodity. The popular method of deriving an answer to this question consists of drawing up tables or graphs showing that prices, after the imposition of a tariff, have risen (or fallen). Of course, this method proves nothing with respect to the effect of the tariff on prices because it does not eliminate the other causes which have contributed to

the price changes. What we desire to know is not how much higher the price will be after the imposition of a tariff than it was before but how much of the increase is due to the tariff and how much to other causes. This is a problem which cannot be solved by "common sense" or "practical" methods; for, as Edgeworth put it, it is "too complicated for the unaided intellect to deal with." Recourse must be had to analytical methods. We therefore turn for guidance to economic theory.

III. SOLUTION PROVIDED BY ECONOMIC THEORY[1]

The teachings of economic theory relative to the effect of a differential tax on the price of the taxed commodity are very definite and point the way to a solution of this problem. They may be stated as follows:[2]

1. The more urgent our demand for the taxed commodity—that is, the more "necessary" the commodity is to us—the more nearly will the price rise to the full amount of the tax.

2. The greater the increase in the quantity of the commodity offered in our market from home (untaxed) sources in consequence of a given change, the less will the price change be.

3. The greater the change in the quantity offered from the taxed sources, the greater will the rise be.

But we have seen that the urgency of our demand for a commodity is measured by the coefficient of the elasticity of demand. Similarly, the change in the quantity supplied in the market in consequence of a given price change is measured by the magnitude of the coefficient of the elasticity of supply. Hence the

[1] This section is reprinted with some changes from the author's "Statistical Measurement of the Elasticity of Demand for Beef," *Journal of Farm Economics,* July, 1924. The author is indebted to Dr. E. G. Nourse, editor of that journal, for permission to reprint.

[2] Cf. A. C. Pigou, "The Known and Unknown in Mr. Chamberlain's Policy," *Fortnightly Review,* January, 1904, p. 44; T. N. Carver, "The Incidence of Costs," *Economic Journal* (December, 1924), and *Principles of National Economy,* chap. xxxii; E. R. A. Seligman, *The Shifting Incidence of Taxation* (4th ed.), pp. 373–79.

foregoing three propositions may be summarized as follows: "Other things being equal, when one source of supply is taxed and the other left free, the rise of price will be greater, the greater are the output and elasticity of the taxed source relatively to the untaxed source."[3]

The validity of this conclusion may, perhaps, be made more obvious by reference to the four cases of Figure 36. Consider first Case A, which is designed to show the effect of a tax when it is levied on the elastic source of supply and when the demand for the commodity is elastic. Let the demand curve be given by DD′. Let there be two sources of supply, 1 and 2. (One of these sources may be foreign and the other domestic or non-dutiable, or both may be foreign.)[4] Let $S_1S'_1$ be the comparatively inelastic supply, and let $S_2S'_2$ be the comparatively elastic supply. If the supply from source 1 were the only one available, the price would be given by the distance that A is above the X-axis. If the supply from source 2 were the only one available, the price would be given by the distance that B is above the X-axis. But both sources of supply are available at the same time. The price will, therefore, depend upon the point where the *aggregated* supply curve crosses the demand curve. The aggregated supply curve may be derived from the supply curves $S_1S'_1$ and $S_2S'_2$ by adding, for each price, the amounts supplied by both sources at that price and taking this sum as the abscissa of the aggregated curve, the ordinate as before representing price. In constructing the aggregated supply curve, the assumption is made, for convenience in exposition, that the quantity produced in source 1 has no effect on the supply schedule of source 2. Thus, when the price is Op, the amount supplied by source 1 is pG and the amount supplied by source 2 is pE; the amount supplied by *both* sources is $pG+pE=pP$. The aggregated supply curve

[3] A. C. Pigou, *Economics of Welfare* (1920), p. 942.

[4] The assumption of only two sources of supply, one relatively more elastic than the other, is made simply for convenience in exposition. The conclusion can be extended to cover any number of sources of supply.

S_2FP crosses the demand curve DD' at P. The market price is, therefore, MP; the amount supplied by source 1, OQ; the

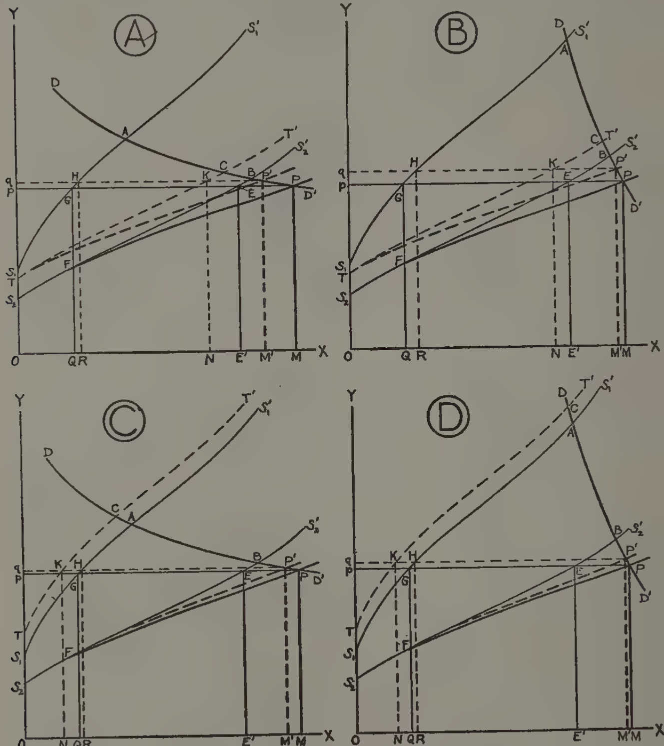

FIG. 36.—Disturbance of the conditions of supply by a tax.
Case A: Tax levied on elastic supply, demand elastic.
Case B: Tax levied on elastic supply, demand inelastic.
Case C: Tax levied on inelastic supply, demand elastic.
Case D: Tax levied on inelastic supply, demand inelastic.

amount supplied by source 2, OE'; and the total amount supplied and demanded is $OQ+OE'$ or OM.

Now let a tax of S_2T units be laid on the (foreign) supply $S_2S'_2$. The supply curve of source 2 becomes TT', and the new aggregated supply curve eventually becomes TP'. The latter crosses the demand curve DD' at P'. The price rises, as a result of the tax, from MP to $M'P'$, or by pq. The total consumption is decreased from OM to OM' or by $M'M$. The supply from the taxed source is reduced by OE' to ON or by NE', but the supply from the untaxed source is increased from OQ to OR, or by QR.

The other effects of the tax, such as, for instance, the changes in the profits of the producers, may also be conveniently studied by means of this diagram. For the present purposes, however, we shall concentrate our attention on the effect of the tariff on prices. We have just seen that the increase in price is measured by pq.

In Case B the conditions of supply are the same as in Case A, but the demand is assumed to be inelastic. It will be seen that the more inelastic the demand, the greater the price change, other things remaining equal.

Cases C and D show the effects of a tariff when it is levied on the comparatively inelastic source of supply, the former showing the effects when the demand is comparatively elastic, and the latter when it is comparatively inelastic. It will be seen that when the tax is levied on the comparatively inelastic source of supply, the increase in price is, other things remaining equal, less than when it is levied on the elastic supply. (Compare Cases C with A, and D with B, with respect to the magnitude of pq.[5])

These conclusions may be expressed in mathematical symbols as follows:[6]

[5] In drawing these graphs it was assumed that enough time has elapsed after the change in the tax for the new equilibrium to be established between supply and demand.

[6] See the following works by A. C. Pigou: (1) "The Known and Unknown in Mr. Chamberlain's Policy," *Fortnightly Review*, January, 1904, p. 44; (2)

Let P be the price in the absence of any tax and let S_D and S_f be the present supplies to the home market from non-dutiable (domestic) and foreign sources, e_D and e_f their respective elasticities of supply, and η the elasticity of the domestic demand curve (η is necessarily negative). Then, if T_f be the tax on the foreign supply, "and if we assume that for the small portion of the demand and supply curves with which we are concerned, the elasticities do not alter," the change in price, ΔP, is given by the formula:

$$\Delta P = T_f \frac{e_f S_f}{e_f S_f + e_D S_D - \eta(S_D + S_f)} . \tag{80}$$

In the foregoing equation, e_D and S_D may be assumed to relate to the aggregate supplies to the home market from both domestic and insular (i.e., from all non-dutiable) sources. If separate values are available for the elasticity of supply of, and the quantity derived from, each separate source, equation (80) becomes somewhat more complex.

Professor Pigou also shows that the same result might be written "in another and, for some purposes, more convenient notation," as follows:

Let A and B represent the domestic (including insular) and foreign *production*, e_D and e_f their respective elasticities of *production*, and η_D and η_f those of demand in the two sources, respectively; then

$$\Delta P = T_f \frac{e_D A - \eta_D A}{e_D A + e_f B - \eta_D A - \eta_f B} . \tag{81}$$

This equation, like the preceding, becomes somewhat more complex when several sources of supply and demand are considered.[7]

Equation (81) differs from equation (80) in two particu-

Protective and Preferential Import Duties (1906), pp. 94–95; (3) *Economics of Welfare* (1920), p. 942. I have taken the liberty to modify some of the symbols used by Professor Pigou.

[7] See especially, Pigou, *Protective and Preferential Import Duties*, pp. 94–95.

lars: first, it allows for the effect of the *foreign demand;* and second, it works with the domestic and the foreign *production* (A and B) and their respective elasticities of production, instead of with the supplies to the home market from non-dutiable (domestic) and foreign sources and their respective elasticities of supply.

Of these differences, the first is the more important, for nearly all commodities which give rise to tariff problems are demanded by more than one country. Furthermore, the *form* of equation (81) is not changed whether we work with production or with supply, although Professor Pigou's distinction between the elasticity of production and the elasticity of supply is conducive to clearness of thought. But since in this study the emphasis has, in general, been more on supply than on production, and since the unnecessary multiplication of symbols is to be avoided, it is desirable to rewrite equation (81) in a slightly different form, or in terms of supply, so as to show more clearly its relation to equation (80), which assumes only one source of demand for the commodity in question. In this form, Professor Pigou's equation (81) becomes

$$\Delta P = T_f \frac{e_f' S_f' - \eta_f D_f}{e_f' S_f' + e_D S_D - \eta_f D_f - \eta_d D_d}$$

$$= T_f \frac{1}{1 + \dfrac{e_D S_D - \eta_d D_d}{e_f' S_f' - \eta_f D_f}}, \qquad (82)$$

where D_d and D_f are the quantities demanded in the home market and abroad; η_d and η_f their respective elasticities of demand; S_f' the *total foreign* supply (and not simply the supply to the home market from foreign sources, which was indicated by S_f in [80]); and e_f' the elasticity of the total foreign supply. The other symbols have the same meaning as in equation (80).[8]

[8] Formula (82) was also deduced by Philip G. Wright, of the Institute of Economics, and, quite independently, by Theodore O. Yntema, of the University of Chicago. I am indebted to these gentlemen for helpful suggestions.

Both formulas are based on the assumption of straight-line demand and

It is evident, therefore, that economic theory offers a satisfactory solution of the problem of effect of a tariff on the price of a commodity, provided we can determine the domestic and foreign supply and demand curves.

IV. EFFECT OF THE TARIFF ON THE PRICE OF SUGAR

The various constants which are called for by formula (82) have either been determined or can be estimated.

The elasticity of the United States demand (η_d) has been found, by four different methods, to be approximately equal to —0.5 under "normal" conditions.[9] That is to say, based on the experience of 1890 to 1914, an increase in the price of sugar of 1 per cent will reduce consumption by ½ of 1 per cent under normal conditions.

The elasticities of the domestic and insular supplies vary according to the price series used[10] and according to whether the supplies to the home market from domestic and insular sources are combined or treated separately. Table XI shows the effect of the different price series on the elasticities of supply. The elasticity of the combined supply from the two sources (e_D) is between $+0.6$ and $+0.8$, or an increase in the price of sugar of 1 per cent will "call forth" an increase of between 0.6 and 0.8 of 1 per cent in the quantities derived from non-dutiable sources.

The only coefficients that have not as yet been determined are the elasticities of the *foreign* demand and supply. We have,

---

supply functions—an assumption which is not contradicted by the empirical supply and demand functions which have been derived for sugar. The quantities appearing in these formulas are "variable constants." When applied to any problem they must be given the values which they assume at the point of equilibrium established *before* any change is made in the duty.

It is perhaps necessary to remark that in any *practical* problem the change in price, ΔP, may be best obtained by the graphic method, which, of course, makes no assumption whatsoever about the nature of the demand and supply curve.

[9] Pp. 45–46, 62–65, 76, 87–90.

[10] Probably the same is also true of the elasticity of demand.

however shown[11] that the elasticities of the total *world* demand and supply, which, of course, include the demand and supply of the United States, are $\eta_w = -0.6$ and $e_w = +0.6$, respectively. As these figures are nearly equal, respectively, to the elasticities of the domestic demand and of the domestic and insular supply, we may assume that even if the United States consumption and the United States production (including imports from insular possessions) had been eliminated, respectively, from the totals

TABLE XI

COMPARISON OF THE ELASTICITIES OF THE SUPPLIES DERIVED FROM DOMESTIC PRODUCTION, IMPORTS FROM INSULAR POSSESSIONS, AND FROM THE TWO SOURCES COMBINED, BY USING DIFFERENT PRICE SERIES

PRICE DATA*	ELASTICITY OF SUPPLY DERIVED FROM		
	Domestic Production e_d	Imports from Insular Possessions e_i	Domestic Production plus Imports from Insular Possessions e_D
Fiscal year	+0.95	+1.05	+0.78
October–May	+0.62	+0.77	+0.58
January–June	+0.62	+0.72	+0.56

* The period covered by the data is from 1903 to 1913. See footnotes to Tables VI, VII, and VIII.

of world demand and supply, the resulting data would still have yielded approximately the same values for the two coefficients as those just given. We may, therefore, to a first approximation, assume the elasticity of the *foreign* demand (i.e., world demand *minus* United States demand) to be $\eta_f = -0.6$, and the elasticity of the *total world-supply* (i.e., world-supply *minus* United States and insular supply) to be $e'_f = +0.6$.[12]

There remain to be determined the quantities that are normally demanded and supplied in the home market and abroad

[11] See p. 168.

[12] A distinction is made between e_f, the elasticity of that part of the foreign (taxed) supply which reaches the United States, and e'_f, the elasticity of the total foreign supply.

(i.e., in the rest of the world). Since these quantities must be the *equilibrium quantities* established before the imposition of the duty, they ought to be determined from the *trends* (or from the averages of the *link relatives*) of the same quantities for the year immediately preceding the change in the duty. For the present purpose, however, we may, with sufficient accuracy, take the *average annual* consumption and production at home and abroad during the five-year period which ended in June, 1914, as the "equilibrium" or "normal" quantities demanded and supplied[13] in the taxed and untaxed sources. These are shown in

TABLE XII

ANNUAL CONSUMPTION AND PRODUCTION OF SUGAR IN THE UNITED STATES AND IN THE REST OF THE WORLD; AVERAGE, 1909–14*

(Millions of Short Tons)

Item	World-Total	United States	Foreign Countries
Demand (consumption)	18.90	3.98	14.92
Supply (production).....	18.90	1.96†	16.94

* Based on Tables 381 and 386 of the *Agricultural Yearbook, 1924.*
† Includes imports from insular possessions.

Table XII. Thus, the "normal" quantities of sugar demanded annually in the home market and abroad during the period under consideration were, respectively, $D_d = 3.98$, and $D_f = 14.92$ millions of short tons; the "normal" quantity supplied annually to the home market from domestic and non-dutiable sources was $S_D = 1.96$ millions of short tons; and the "normal" total foreign supply was $S'_f = 16.94$ millions of short tons per annum. These figures are the *weights* that must be applied to the coefficients of the elasticities of supply and demand in formula (82).

Table XIII brings together the values that have been assigned to the eight constants of this formula.

[13] It should be kept in mind that the supply curves were based on data for the years 1903–13.

Substituting these values in (82), we obtain

$$\Delta P = T_f \frac{1}{1+\frac{0.6(1.96)+0.5(3.98)}{0.6(16.94)+0.6(14.92)}} = 0.86\, T_f \qquad (83)$$

or the increase in price is 86 per cent of the duty. That is, if during 1909–13 the duty on sugar had been increased or decreased by 1 cent per pound, the normal effect would have been to increase or decrease the price per pound by 0.86 of one cent. The assumption is made, of course, that sufficient time has elapsed after the change in the duty for the new equilibrium to be established between supply and demand.

TABLE XIII

ELASTICITIES OF DEMAND AND SUPPLY OF SUGAR IN THE UNITED STATES AND ABROAD, AND THE AVERAGE ANNUAL DEMAND AND SUPPLY DURING 1909–14

Item	United States		Foreign Countries	
	Symbol	Value	Symbol	Value
Elasticity of demand	η_d	−0.5	η_f	− 0.6
Elasticity of supply	e_D	+0.6*	e'_f	+ 0.6
Quantity demanded (millions of short tons)	D_d	3.98	D	14.92
Quantity supplied (millions of short tons)	S_D	1.96†	S'_f	16.94

* Relates to United States production and imports from insular possessions.

† Includes imports from insular possessions.

In applying formula (82), two interesting questions present themselves: first, should not the figures of those countries which consume all that they produce be eliminated from the totals for "foreign countries," because this production does not affect the world-price?; second, if this is done, would not the result be a significantly lower figure for the increase in price due to the tariff? The answer is that there is no self-contained economy with respect to sugar whose production is of sufficient importance to affect the world-price. Even if the entire output of

India,[14] whose production is second only to that of Cuba, be eliminated from the totals for the foreign countries (see Table XII), the effect would only be to reduce ΔP in equation (83) from 0.86 T_f to 0.83 T_f.

Another question that presents itself relates to the accuracy of the elasticity of the domestic supply. While four different methods give practically identical values for the elasticity of the United States demand, the same is not true of the elasticity of supply. As has already been pointed out, the elasticities of the domestic and insular supplies vary according to the price series used and according as the supplies to the home market from United States and insular sources are combined or treated separately. The elasticity of the combined supply from the two sources (e_D) is between +0.6 to +0.8, approximately, depending upon the price series used (see Table XI). In the foregoing computations, e_D was given the lower value, or +0.6. Would not the assumption that the elasticity of the combined supply is nearer to the upper limit, or +0.8, lead to a radically different conclusion as to the effect of a tariff on the price of sugar? By substituting different values for e_D in equation (82), we can easily see that even if the elasticity of the domestic supply be taken as high as +1.0, this would have only a relatively small effect on ΔP, or a decrease in its value from $+0.86T_f$ to $+0.83T_f$.

We conclude, therefore, that under such average conditions of demand and supply as had prevailed during the five years before the war, the increase in price due to a tariff on sugar would be approximately 86 per cent of the duty, the remaining 14 per cent being borne by the foreign exporters. *Whether the present tariff on sugar has a greater or less effect on price than it would have had during the five years before the war, depends on whether the ratio of the output and elasticity of the taxed supply to the output and elasticity of the untaxed (domestic and insular) supply is greater or less than it was during 1909–13.*

[14] India's average annual production for the period from 1909–10 to 1913–14 was 2,650 million short tons. See *Agriculture Yearbook, 1924,* Table 386.

Formulas and methods similar to those employed above may also be derived for estimating the probable effect of any tariff on imports, domestic production and consumption, and the revenue derived by the government. Such information would not point the way to a "right," or "correct," tariff, as such terms are incapable of definition. It would, however, indicate the *probable consequences* of the present or proposed tariff, and the duty-levying agency would be in a position to weigh and balance the advantages of any tariff against its disadvantages.

It is, perhaps, unnecessary to add that the use of the methods herein discussed presupposes a general knowledge of the industry or commodity under consideration.

V. THE ECONOMIC THEORY OF THE "FLEXIBLE-TARIFF" PROVISION

This conclusion naturally invites comparison with the theory underlying section 315 of the Tariff Act of 1922, which empowers the president with certain limitations to change duties by proclamation. This has come to be known as the "flexible-tariff" provision. The substance of it is that when the president finds upon investigation that any duty contained in this Act does not "equalize" the difference in the cost of production in the United States and the principal competing country, he shall raise or lower the duty by an amount necessary to "equalize" the difference provided that no change in the rate of duty shall exceed 50 per cent, except in the special case when the basis of assessing the rate of duty may be changed from the foreign value of the article concerned to the American selling-price. The president is, however, specifically denied authority to transfer an article from the free list to the dutiable list, or from the dutiable list to the free list. The Act further provides that the investigations to assist the president in ascertaining the difference in cost of production shall be made by the United States Tariff Commission and that he shall issue no proclamation changing a duty until an investigation has been made.

This cost-of-production formula for tariff-making has been vigorously attacked by several authorities. Thus a former chairman of the United States Tariff Commission, Dr. Thomas Walker Page, argues[15] that—

(1) The difference in costs of production is unfit for general application as a measure of duties because it would make some duties too low and others altogether too high. . . . .[16]

(2) The application of the difference between costs as a measure of duties is usually impossible owing to the difficulty of finding what the difference amounts to. . . . .[17]

(3) Even if cost data were reliable the fact would remain that there is in every country more than one cost at which any commodity is produced in commercial quantities. . . . .[18]

(4) The difficulties in obtaining and verifying costs of production are infinitely greater in foreign countries than they are in the United States. . . . .[19]

He concludes that—

. . . . it is rarely possible to ascertain accurately the difference in costs of production at home and abroad. To use as the basis of a general tariff act a thing so fleeting, evasive, or shadowy would be neither right nor possible.[20]

Professor Taussig, the first chairman of the United States Tariff Commission, is equally vigorous in his denunciation of this quasi-automatic formula,[21] though on slightly different grounds.

With these criticisms, however, we shall not concern ourselves. In fact, we shall assume for the present purpose that it is entirely possible and practicable to obtain adequate and unequivocal cost data at home and abroad; that there are no difficulties of any importance connected with joint costs, original investment, entrepreneurs' wages as distinguished from his profits, market boundaries, transportation costs, size of sample, etc.;

[15] Thomas Walker Page, "Tariff Making by Formula," *Making the Tariff in the United States,* chap. vi.

[16] *Ibid.,* p. 74.

[17] *Ibid.,* p. 83.

[18] *Ibid.,* p. 89.

[19] *Ibid.,* p. 95.

[20] *Ibid.,* p. 99.

[21] F. W. Taussig, "The United States Tariff Commission and the Tariff," *American Economic Review,* XVI, No. 1 Supplement (March 1926), 170–81.

and we shall address ourselves to certain theoretical aspects of the cost-of-production formula which have not, perhaps, received the attention that they deserve.

1. A good many, if not most, of the officials and economists who work with the accounting cost curve (particular expenses curve) that has been made familiar through the publications of the United States Tariff Commission, the Federal Trade Commission, and other bodies, are under the impression that it is the statistical equivalent of the theoretical cost-of-production curve that is discussed in works on economic theory. This leads them to the tacit assumption that price is determined at the point where this particular expenses curve (ogive) is crossed by the demand curve; and this assumption is, in turn, responsible for fallacious conclusions based on the "bulk-line" principle.

The "bulk-line" practice originated with the Price-Fixing Committee of war-time fame.[22] This Committee, having been called upon to fix the prices of commodities for which the war-time demand was particularly inelastic, adopted the course of fixing—

> . . . . a maximum price, the same for public purchases and for private, the government taking all it wanted and the general public scrambling to buy at the maximum prices whatever it could get.[23]
>
> The basis upon which the findings and agreements of the Price-Fixing Committee rested was in the main the familiar and plausible one of cost of production. . . . . The figures and statistics of cost which were utilized were supplied in all cases by the Federal Trade Commission.[24]

That is to say, the figures and statistics related to what we have termed "accounting costs."

> In general, in the price-fixing operations for such articles, regard was had to the marginal producers. A new jargon was often used, superseding that of the economists: the "bulk-line" producer and "bulk-line" cost spoken of. It was cost of production at the hands of this marginal or bulk-line person that usually formed the basis of the prices fixed. Sporadic cases of

[22] See F. W. Taussig, "Price-Fixing as Seen by a Price-Fixer," *Quarterly Journal of Economics,* XXXIII (February, 1919), 205–41.

[23] *Ibid.,* p. 216. [24] *Ibid.*

exceptionally and extremely high cost were disregarded, and properly so. Extreme high costs in individual cases are part of the flotsam and jetsam of economic life—accidents, of no real significance. The marginal or price-determining producer was found at the point where from 80 per cent to 90 per cent of the output was included. As a rule a price was fixed which would "protect" four-fifths or nine-tenths of the entire output.[25]

As a war-time measure this procedure cannot be criticized. Some simple, easily-understood method had to be adopted which would meet the needs of the Committee, and the "bulk-line" method was probably as good as any. But the authority of the eminent economists who served on the Price-Fixing Committee is being used in support of the same method of computing costs for tariff making, without a critical examination of the implicit assumptions involved.

To judge from the current attempts to apply the "bulk-line" method to tariff problems, the reasoning of some of the "bulk-line" adherents seems to run something like this:

If the demand curve were available we could, by bringing it into juxtaposition with the cost curve, determine the price of the commodity and the quantity produced at a profit (or loss). But notwithstanding our ignorance of the shape of the demand curve, we know the ordinate of the point at which this curve crosses the supply curve, for that ordinate is equal to the *price*. By drawing a line—a bulk-line—parallel to the X-axis and at a perpendicular distance above it equal to the price (i.e., ordinate) and projecting this line until it cuts the cost curve, we can also obtain the abscissa of the point at which the cost curve is crossed by the demand curve, and thus determine the effects of a given duty.

The entire argument is fallacious:

The "cost" curve used in the "bulk-line" method is not a true (theoretical) cost curve but is what we have termed an "accounting cost curve," or Marshall's "particular expenses curve," or simply the frequency distribution of costs for one year cumulated upward. The true cost curve would show changes in annu-

[25] *Ibid.*, p. 219.

al (or periodic) output, i.e., changes in the *scale of production,* corresponding to changes in unit costs. The accounting cost curve shows nothing of the kind.[26] It cannot, therefore, be safely used to determine the effect of a given duty on the conditions of supply.

2. Even if it were possible directly to deduce the theoretical cost curve, it could not always be brought into juxtaposition with the demand curve; for price is determined not at the point where the demand curve crosses the cost curve, but where it crosses the supply curve. Under conditions of increasing costs, the unit supply curve is the curve of marginal costs and it lies above the curve of average unit costs. (Compare *sa′* and *sa,* Fig. 27*B*.) Under conditions of decreasing costs, the curve of marginal costs is below the curve of average costs, *but then the supply curve must be taken as the curve of average unit costs.*[27] (Compare *sa* and *sa′*, Fig. 27*D*.) Under conditions of constant costs the curve of marginal costs coincides with the curve of average costs. This means, of course, that under conditions of constant cost, there is no difference between the cost curve and the supply curve. (See Fig. 27*F*.) However, there are very good reasons for believing that the two conditions under which the cost curve would be taken as the supply curve are inconsistent with the assumption of free competition. There is, then, in theory, only one type of supply curve which is in the long run compatible with the assumption of free competition—the positively inclined (unit) supply curve. It does *not* coincide with the (unit) cost curve. To confuse the two curves may lead to erroneous conclusions as to the effect of the tariff on conditions of supply.

3. Scientifically to determine the effect of a tariff on conditions of supply, we must work, not with cost curves, but with supply and demand curves.

[26] For a fuller discussion of the difference between the accounting cost curve and the theoretical cost curve, see *supra,* pp. 104–11.

[27] See p. 120, above, and Irving Fisher, *Elementary Principles of Economics,* pp. 315–16.

4. Modern statistical methods enable us to deduce concrete statistical supply and demand curves for a good many commodities, and consequently to measure the effect of a given duty on the price, domestic production, and imports of the article concerned—something that could not be done by the "bulk-line" method.

In view of the foregoing, is it not clear that the more scientific approach to the tariff problem is through the methods suggested in this chapter, which take into consideration the forces behind both demand and supply rather than through the cost or "bulk-line" method? The supply curve has several advantages over the cost curve. First, it overcomes theoretical objections to the cost method mentioned above; and second, it does not lead us into the metaphysics of farm costs. All it tells us is the relation between a given change in price (not cost) and the corresponding change in supply. It is a summary presentation in quantitative terms of how producers react to the price stimulus. It does not inquire why one producer will react one way and another producer in another way and then make their reactions comparable by imputing certain prices to the invested capital, labor, etc., as is done by the cost method. It simply shows what is the average reaction of producers to a given change in price. That is, it summarizes in a sort of "mental shorthand" the whole mesh of environing conditions affecting supply.

Finally, it must be emphasized that the possibility of deducing concrete, statistical demand and supply curves will not only enable us to measure the effect of a given duty on conditions of supply, but will also enable us successfully to attack other problems which are at present considered insoluble.

APPENDIX I

COMMENTS ON PROFESSOR LEHFELDT'S METHOD OF DERIVING THE ELASTICITY OF DEMAND FOR WHEAT

Instead of first finding the law of demand for wheat and then deducing the elasticity of demand, Professor Lehfeldt removes the trends from the statistics of supply and prices, finds the standard deviations of the corrected series (or, rather, of the *logarithms* of the corrected series), and computes the coefficient of elasticity of demand from the formula:

$$\sigma_x/\sigma_y \tag{I}$$

where σ_x is the standard deviation of the *logarithms* of the crops (supply) during the period under consideration, and σ_y is the standard deviation of the *logarithms* of the corresponding prices.[1] He finds that $\sigma_x = 26.4$, $\sigma_y = 43$, and concludes accordingly that $\eta = 26.4/43 = 0.6$.

This procedure is open to the objection that it assumes the existence of perfect correlation between the variables when, by his own computations, the coefficient of correlation is $r = -0.44$. And when this moderate value of r is taken into consideration, it is not at all clear that the elasticity of demand for wheat is $\eta = -0.6$. A closer examination of Lehfeldt's method will throw some light on its *rationale*.

Professor Lehfeldt refers to 0.6 as being *the* elasticity of demand for wheat. This involves the assumption that the law of demand for wheat is a hyperbola of the form

$$xy^n = c \tag{II}$$

otherwise the coefficient of elasticity of demand would vary from point to point on the demand curve. As he works with logarithms, equation (II) may be written as

$$\log x + n \log y = \log c \tag{IIa}$$

or, denoting the logarithms by capital letters and transposing,

$$Y = -mX + B \tag{IIb}$$

[1] We have taken the liberty of changing Lehfeldt's notation so as to conform to that used in this study. See *Economic Journal* (1914), pp. 212 ff.

where $m=\frac{1}{n}$ is the slope, and $B=\frac{1}{n}\log c$ is the Y-intercept. When (IIb) is fitted to the data we may select for our demand curve either the regression of Y on X:

$$Y'=r\frac{\sigma_y}{\sigma_x}X' \tag{III}$$

or the regression of X on Y:

$$X'=r\frac{\sigma_x}{\sigma_y}Y' \tag{IIIa}$$

the accents indicating that the variables are measured from their respective means. Since the coefficient of elasticity of demand is, by definition,

$$\eta=\frac{dx}{x}\bigg/\frac{dy}{y}=\frac{d\log x}{d\log y}=\frac{dX'}{dY'} \tag{IV}$$

it is clear that we shall obtain two different values for $\eta=\frac{dX'}{dY'}$, depending upon whether we choose to derive it from (III) or (IIIa).

From III,

$$\eta=\frac{dX'}{dY'}=1\div\frac{dY'}{dX'}=\frac{\sigma_x}{r\sigma_y}=\frac{26.4}{(-0.44)(43.0)}=-1.4\,. \tag{V}$$

From IIIa,

$$\eta=\frac{dX'}{dY'}=r\frac{\sigma_x}{\sigma_y}=\frac{(-0.44)(26.4)}{43.0}=-0.27. \tag{Va}$$

It will be seen from the foregoing equations that Professor Lehfeldt's formula for the coefficient of elasticity of demand (i.e., $\eta=\sigma_x/\sigma_y$) is true only when r is numerically equal to 1.0.

We do not wish to be understood as saying that Professor Lehfeldt's figure of 0.6 is not a good first approximation to the elasticity of demand for wheat. Though the "line of best fit" gives a value for η of only -0.36, we cannot say definitely that it is a more probable value than Lehfeldt's figure because we are not acquainted with the nature of his data. Furthermore, we are in thorough agreement with him when he says, ". . . . the roughest attempt to measure a coefficient of elasticity would be better than none." All that we wish to point out here is that one cannot derive a satisfactory value of the elasticity of demand without first deriving the most probable demand curve.

APPENDIX II

TABLES RELATING TO DEMAND

TABLE I

THE TOTAL CONSUMPTION AND THE MONEY PRICES OF SUGAR IN THE UNITED STATES, AND THE LINK RELATIVES OF CONSUMPTION AND PRICES, 1890–1914

Year Beginning January 1	Total Consumption (in Thousands of Short Tons) x	New York Wholesale Price of Refined Sugar (Yearly Averages in Cents per Lb.) y	Link Relatives of Consumption X	Link Relatives of Prices Y
1890	1,654	6.171	....	....
1891	2,116	4.641	1.279	0.752
1892	2,076	4.346	0.981	0.936
1893	2,136	4.842	1.029	1.114
1894	2,254	4.120	1.055	0.851
1895	2,184	4.152	0.969	1.008
1896	2,195	4.532	1.005	1.092
1897	2,319	4.503	1.056	0.994
1898	2,243	4.965	0.967	1.103
1899	2,327	4.919	1.037	0.991
1900	2,486	5.320	1.068	1.082
1901	2,657	5.050	1.069	0.949
1902	2,874	4.455	1.082	0.882
1903	2,856	4.638	0.994	1.041
1904	3,099	4.772	1.085	1.029
1905	2,948	5.256	0.951	1.101
1906	3,208	4.515	1.088	0.859
1907	3,353	4.649	1.045	1.030
1908	3,568	4.957	1.064	1.066
1909	3,649	4.765	1.023	0.961
1910	3,752	4.972	1.028	1.043
1911	3,754	5.345	1.001	1.075
1912	3,925	5.041	1.046	0.943
1913	4,192	4.278	1.068	0.849
1914	4,212	4.683	1.005	1.095

TABLE I A

THE PER CAPITA CONSUMPTION AND THE REAL PRICES OF SUGAR IN THE UNITED STATES, AND THE LINK RELATIVES OF CONSUMPTION AND PRICES, 1890–1914

Year Beginning January 1	Per Capita Consumption (in Pounds) x	Real Price Per Pound* (in Cents) y	Link Relatives of Per Capita Consumption X	Link Relatives of Real Price Y
1890	52.8	6.643	....	....
1891	66.3	5.061	1.256	0.762
1892	63.8	5.053	0.962	0.998
1893	64.4	5.484	1.009	1.085
1894	66.7	5.209	1.036	0.950
1895	63.4	5.171	0.951	0.993
1896	62.5	5.901	0.986	1.141
1897	64.8	5.863	1.037	0.994
1898	61.5	6.183	0.949	1.055
1899	62.6	5.720	1.018	0.925
1900	65.2	5.727	1.042	1.001
1901	68.7	5.574†	1.054	0.973
1902	72.8	4.626	1.060	0.830
1903	70.9	4.703	0.974	1.017
1904	75.3	4.840	1.062	1.029
1905	70.5	5.331	0.936	1.101
1906	76.1	4.422	1.079	0.829
1907	77.5	4.313	1.018	0.975
1908	81.2	4.803	1.048	1.114
1909	81.8	4.285	1.007	0.892
1910	81.6	4.294	0.998	1.002
1911	79.2	5.009	0.971	1.167
1912	81.3	4.441	1.027	0.887
1913	85.4	3.730	1.050	0.840
1914	84.3	4.166	0.987	1.117

* Real price = money price divided by Bureau of Labor Statistics index number of wholesale prices "all commodities," average 1900–1909 = 1.00.

† Throughout the computation the figure 5.436 was erroneously used in place of this one. The effect on the result is negligible.

TABLE II

THE TREND RATIOS OF THE TOTAL CONSUMPTION AND THE MONEY PRICES OF SUGAR IN THE UNITED STATES, 1890–1914

Year Beginning January 1	Total Consumption (in Thousands of Short Tons) x	Ordinates of the Secular Trend of Consumption* T_x	Consumption Ratios $X=x/T_x$	Wholesale Price per Pound at New York (in Cents) y	Ordinates of the Secular Trend of Prices† T_y	Money Price Ratios $Y=y/T_y$
1890	1,654	1,923	0.860	6.171	5.248	1.176
1891	2,116	1,954	1.083	4.641	5.033	0.922
1892	2,076	1,994	1.041	4.346	4.862	0.894
1893	2,136	2,041	1.047	4.842	4.732	1.023
1894	2,254	2,097	1.075	4.120	4.639	0.888
1895	2,184	2,159	1.012	4.152	4.579	0.907
1896	2,195	2,229	0.985	4.532	4.548	0.996
1897	2,319	2,305	1.006	4.503	4.542	0.991
1898	2,243	2,388	0.939	4.965	4.558	1.089
1899	2,327	2,476	0.940	4.919	4.590	1.072
1900	2,486	2,569	0.968	5.320	4.636	1.148
1901	2,657	2,668	0.996	5.050	4.692	1.076
1902	2,874	2,771	1.037	4.455	4.753	0.937
1903	2,856	2,878	0.992	4.638	4.815	0.963
1904	3,099	2,989	1.037	4.772	4.876	0.979
1905	2,948	3,104	0.950	5.256	4.930	1.066
1906	3,208	3,222	0.996	4.515	4.974	0.908
1907	3,353	3,343	1.003	4.649	5.004	0.929
1908	3,568	3,466	1.029	4.957	5.016	0.988
1909	3,649	3,590	1.016	4.765	5.007	0.952
1910	3,752	3,717	1.009	4.972	4.971	1.000
1911	3,754	3,844	0.977	5.345	4.906	1.089
1912	3,925	3,973	0.988	5.041	4.807	1.049
1913	4,192	4,102	1.022	4.278	4.671	0.916
1914	4,212	4,231	0.996	4.683	4.493	1.042

* The equation to the secular trend of total consumption is $T_x=2770.84+105.318t+2.1242t^2-0.06352t^3$. Origin at July 1, 1902.

† The equation to the secular trend of the money prices is $T_y=4.75281+0.062466t+0.0008192t^2-0.0006522t^3$. Origin at July 1, 1902.

TABLE II A

THE TREND RATIOS OF THE PER CAPITA CONSUMPTION AND REAL PRICES OF SUGAR IN THE UNITED STATES, 1890–1914

Year Beginning January 1	Per Capita Consumption (in Pounds) x	Ordinates of the Secular Trend of Consumption* T_x	Consumption Ratios $X=x/T_x$	Real Price Per Pound† (in Cents) y	Ordinates of the Secular Trend of Prices‡ T_y	Real Price Ratios $Y=y/T_y$
1890	52.8	61.0	0.866	6.643	5.581	1.190
1891	66.3	61.0	1.087	5.061	5.622	0.900
1892	63.8	61.2	1.042	5.053	5.645	0.895
1893	64.4	61.6	1.045	5.484	5.653	0.970
1894	66.7	62.2	1.072	5.209	5.645	0.923
1895	63.4	62.8	1.010	5.171	5.624	0.919
1896	62.5	63.6	0.983	5.901	5.590	1.056
1897	64.8	64.5	1.005	5.863	5.545	1.057
1898	61.5	65.6	0.938	6.183	5.489	1.126
1899	62.6	66.7	0.939	5.720	5.424	1.055
1900	65.2	67.8	0.962	5.727	5.350	1.070
1901	68.7	69.0	0.996	5.574§	5.270	1.058
1902	72.8	70.3	1.036	4.626	5.183	0.893
1903	70.9	71.6	0.990	4.703	5.092	0.924
1904	75.3	73.0	1.032	4.840	4.996	0.969
1905	70.5	74.3	0.949	5.331	4.899	1.088
1906	76.1	75.6	1.007	4.422	4.799	0.921
1907	77.5	77.0	1.006	4.313	4.700	0.918
1908	81.2	78.3	1.037	4.803	4.601	1.044
1909	81.8	79.5	1.029	4.285	4.504	0.951
1910	81.6	80.7	1.011	4.294	4.410	0.974
1911	79.2	81.8	0.968	5.009	4.320	1.159
1912	81.3	82.9	0.981	4.441	4.235	1.049
1913	85.4	83.8	1.019	3.730	4.157	0.897
1914	84.3	84.6	0.996	4.166	4.086	1.020

* The equation to the secular trend of the per capita consumption of sugar is $T_x=70.324+1.2952t+0.01729t^2-0.002146t^3$. Origin at July 1, 1902.

† Real price = money price divided by Bureau of Labor Statistics index number of wholesale prices "all commodities," average 1900–1909 = 1.00.

‡ The equation to the secular trend of the real price of sugar is $T_y=5.18299-0.089145t-0.0024274t^2+0.00018626t^3$. Origin at July 1, 1902.

§ Throughout the computation the figure 5.436 was erroneously used in place of this one. The effect on the result is negligible.

APPENDIX III

TABLES RELATING TO SUPPLY

TABLE I

Sugar: Production, Trade, and Consumption of Continental United States, 1902–14

(Source: Adapted from *Agricultural Yearbook, 1924*, p. 802)

Year Beginning July	Production* (1,000 Short Tons)	Brought from Insular Possessions† (1,000 Short Tons)	Net Imports from Foreign Countries‡ (1,000 Short Tons)	Domestic Exports§ (1,000 Short Tons)	Consumption	
					Total†† (1,000 Short Tons)	Per Capita (Pounds)
1902......	591.30	509.87	2,096.78	7.10	3,190.85	79.6
1903......	518.67	528.64	1,809.98	9.82	2,847.48	69.6
1904......	657.10	591.01	1,800.42	10.74	3,037.79	72.8
1905......	703.52	613.26	1,952.29	13.26	3,255.81	76.6
1906......	755.77	627.16	2,179.15	14.84	3,547.24	81.9
1907......	857.86	792.59	1,663.74	17.00	3,297.20	74.7
1908......	840.28	797.48	2,051.56	44.61	3,644.71	81.1
1909......	882.63	927.75	1,934.75	72.38	3,672.75	80.3
1910......	903.48	943.70	1,845.27	36.60	3,655.85	78.6
1911......	1,005.34	1,187.66	1,832.42	50.38	3,475.04	84.2
1912......	907.07	1,026.97	2,266.42	30.96	4,169.50	87.0
1913......	1,088.94	936.38	2,463.25	37.19	4,451.38	91.6
1914......	1,022.83	1,098.31	2,529.96	302.64	4,348.46	88.2

* Predominately raw except beet-sugar production and domestic exports which are chiefly refined; 1909-to-date production and domestic exports converted to raw.

† From Hawaii, Porto Rico, and Philippine Islands (Virgin Islands included, 1917 and subsequently).

‡ Cuba included. Philippine Islands excluded 1900 and subsequently.

§ Shipments to Hawaii and Porto Rico included.

†† Consumption for all purposes. No account taken of stocks at beginning or end of year.

TABLE I A

LINK RELATIVES OF THE PRODUCTION, TRADE, AND CONSUMPTION OF SUGAR IN THE UNITED STATES, 1903–14

Year Beginning July	Link Relatives of— Domestic Production X_1	Imports from Insular Possessions X_2	Domestic Production Plus Imports from Insular Possessions X_3	Net Imports from Foreign Countries X_5	Total Consumption X_6	Per Capita Consumption X_7
1902*	1.077	....	1.093	....	....	....
1903	0.877	1.036	0.951	0.863	0.892	0.874
1904	1.267	1.118	1.191	0.995	1.067	1.046
1905	1.071	1.038	1.055	1.084	1.072	1.052
1906	1.074	1.022	1.050	1.116	1.089	1.069
1907	1.135	1.264	1.193	0.763	0.930	0.912
1908	0.980	1.006	0.992	1.233	1.106	1.086
1909	1.050	1.164	1.105	0.943	1.008	0.990
1910	1.024	1.017	1.020	0.954	0.995	0.979
1911	1.113	1.259	1.187	0.993	1.087	1.071
1912	0.902	0.865	0.882	1.237	1.049	1.033
1913	1.201	0.912	1.047	1.087	1.067	1.053
1914*	0.939	1.173	1.047	1.027	0.977	0.963

* The figures for this year were not used in the correlation of synchronous series.

TABLE II

SUGAR, GRANULATED: AVERAGE WHOLESALE PRICE PER POUND, NEW YORK, 1902–14

(Source: *Agricultural Yearbook, 1924*, p. 811)

Compiled from Bureau of Labor Statistics reports.

Year	Price, in Cents Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Average
1902	4.5	4.5	4.5	4.5	4.5	4.4	4.4	4.4	4.4	4.4	4.4	4.6	4.5
1903	4.6	4.6	4.6	4.7	4.7	4.7	4.8	4.8	4.8	4.6	4.5	4.4	4.6
1904	4.3	4.3	4.4	4.4	4.7	4.8	4.9	5.0	5.0	4.8	5.2	5.5	4.8
1905	5.8	5.9	5.9	5.9	5.7	5.5	5.1	5.1	4.8	4.5	4.4	4.5	5.3
1906	4.4	4.3	4.4	4.4	4.4	4.4	4.6	4.7	4.7	4.6	4.6	4.6	4.5
1907	4.6	4.5	4.6	4.6	4.8	4.9	4.8	4.7	4.6	4.6	4.6	4.6	4.7
1908	4.7	4.6	5.0	5.3	5.3	5.2	5.2	5.0	5.0	4.8	4.6	4.5	4.9
1909	4.5	4.4	4.6	4.8	4.8	4.7	4.7	4.8	4.9	4.9	5.0	4.9	4.8
1910	4.9	4.9	5.2	5.1	5.2	5.0	5.1	5.1	5.0	4.8	4.6	4.7	5.0
1911	4.7	4.6	4.7	4.7	4.8	4.9	5.1	5.7	6.6	6.6	6.1	5.6	5.3
1912	5.4	5.5	5.5	5.1	4.9	5.0	4.9	4.9	5.0	4.8	4.8	4.8	5.0
1913	4.5	4.2	4.2	4.1	4.1	4.1	4.5	4.6	4.5	4.2	4.2	4.1	4.3
1914	3.9	3.9	3.8	3.7	4.0	4.2	4.2	6.5	6.8	5.9	4.9	4.8	4.7

TABLE II A

SUGAR, GRANULATED: AVERAGE WHOLESALE PRICE PER POUND. NEW YORK, 1902–14

(Absolute and Relative Prices)

YEAR	AVERAGE FOR—			LINK RELATIVES FOR—		
	Year Beginning July 1 (Cents) y_1	Eight Months Beginning Oct. 1 (Cents) y_2	Six Months Beginning January 1 (Cents) y_3	Year Beginning July 1 Y_1	Eight Months Beginning Oct. 1 Y_2	Six Months Beginning Jan. 1 Y_3
1902	4.54	4.58	4.48	....	....	....
1903	4.57	4.45	4.65	1.006	0.972	1.038
1904	5.42	5.59	4.48	1.188	1.256	0.963
1905	4.56	4.41	5.78	0.840	0.789	1.290
1906	4.65	4.61	4.38	1.020	1.045	0.758
1907	4.83	4.84	4.67	1.039	1.050	1.066
1908	4.74	4.62	5.02	0.981	0.955	1.075
1909	4.96	5.01	4.63	1.046	1.084	0.922
1910	4.81	4.70	5.05	0.970	0.938	1.091
1911	5.59	5.59	4.73	1.163	1.189	0.937
1912	4.53	4.44	5.23	0.811	0.794	1.106
1913	4.13	3.98	4.20	0.912	0.896	0.803
1914*	5.57	5.42	3.92	1.347	1.362	0.933

* The figures for this year were not used in the correlation of synchronous series.

TABLE III

DATA FOR COMPUTING THE WORLD DEMAND AND SUPPLY CURVES FOR SUGAR, THE ANNUAL WORLD-PRODUCTION OF SUGAR, AND THE AVERAGE ANNUAL NEW YORK WHOLESALE PRICES OF GRANULATED SUGAR, 1902–3 TO 1913–14

I Year	II World Production* (Thousands of Short Tons) x	III New York Wholesale Prices of Granulated Sugar† (Cents per Pound) y	IV Relative Production Xw	V Relative Price Y_1	VI Relative Production of Current Year Xw	VII Relative Price of Preceding Year Y_1
1902–3	13,247	4.54	....	....	....	....
1903–4	13,659	4.57	1.031	1.006	....	....
1904–5	13,084	5.42	0.958	1.188	0.958	1.006
1905–6	15,621	4.56	1.194	0.840	1.194	1.188
1906–7	16,210	4.65	1.038	1.020	1.038	0.840
1907–8	15,591	4.83	0.962	1.039	0.962	1.020
1908–9	16,311	4.74	1.046	0.981	1.046	1.039
1909–10	16,708	4.96	1.024	1.046	1.024	0.981
1910–11	19,021	4.81	1.138	0.970	1.138	1.046
1911–12	17,793	5.59	0.935	1.163	0.935	0.970
1912–13	20,394	4.53	1.146	0.811	1.146	1.163
1913–14	20,907	4.13	1.025	0.912	1.025	0.811
Mean	....	....	1.0452	0.9978	1.0466	1.0064

* Loose-leaf service, Truman G. Palmer, *Concerning Sugar*, p.C^{1-D}. A note explains that the production figures are converted from Willett and Gray, long tons. The definition of "year" is not given in the table. A comparison of these figures with the corresponding estimates for production given in the *Yearbook of the Department of Agriculture, 1924*, p. 808, suggests that the term relates to the "crop year" for the countries in which the sugar season begins in the autumn months and is completed in the following year, except in the case of cane-sugar-producing countries where the season begins in May or June and is completed in the same calendar year.

For the purpose of comparing production with prices the world "crop year" is assumed to begin in July. The harvesting-period of sugar-producing countries is given by Palmer, *op. cit.*, p. C (second sheet).

† Average for year beginning July. *Yearbook, 1924*, p. 811.

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